Chapter 4

Fermat's and Wilson's Theorems

Introduction

One way to investigate problems in number theory is to look at special cases of the problem and to draw general conclusions from the empirical evidence accumulated. You have already seen instances, and you will see many more, of dangers in this approach when seemingly clear patterns of behaviour can go unexpectedly wrong. Consider the numbers $2^n - 2$, for positive integers n. Investigation of the prime decompositions of these numbers would yield that $2^2 - 2$ is divisible by 2, $2^3 - 2$ is divisible by 3, $2^5 - 2$ is divisible by 5 and $2^7 - 2$ is divisible by 7. On the other hand, taking composite exponents, $2^4 - 2$ is not divisible by 4 and $2^6 - 2$ is not divisible by 6. It is beginning to look as if $2^n - 2$ is divisible by n when n is prime but not when it is composite. Trying more and more values of n, we find that $2^{17} - 2$ is divisible by 17, $2^{18} - 2$ is not divisible by 18, $2^{19} - 2$ is divisible by 19, and this no doubt fortifies our faith in the truth of this result. But no matter how much numerical evidence is gathered, we cannot be sure that the pattern really does persist until we have a general proof.

For many hundreds of years, mathematicians believed the result that $2^p - 2$ is divisible by p if, and only if, p is prime. They had good evidence to support their conjecture, but not a proof. It was 1819 before a counter-example was discovered: $341 = 11 \times 31$ is not prime but $2^{341} - 2$ is divisible by 341.

In the seventeenth century, Fermat showed interest in these numbers and he extended his researches beyond just looking at powers of 2. His interest was captured by numbers $a^p - a$ for any integer a, and he succeeded in proving that if p is prime then $a^p - a$ is divisible by p. He offered no conclusions when the exponent is composite.

You can learn more about Fermat's life in the *History Reader* for this topic.

In his book *Men of Mathematics* (1937, Pelican), E.T. Bell writes:

> It is difficult if not impossible to state why some theorems in arithmetic are considered 'important' while others, equally difficult to prove, are dubbed trivial. One criterion, although not necessarily conclusive, is that the theorem should be of use in other fields of mathematics. Another is that it should suggest researches in arithmetic or in mathematics generally, and a third that it should be in some respect universal. Fermat's Theorem satisfies all of these somewhat arbitrary demands: it is of indispensable use in many departments of mathematics, including the theory of groups, which in turn is at the root of the theory of algebraic equations; it has suggested many investigations, of which the entire subject of primitive roots may be recalled to mathematical readers as an important instance; and finally it is universal in that it states a property of *all* prime numbers – such statements are extremely difficult to find and very few are known.

When, in the first section of this chapter, we meet Fermat's Theorem, which tends to be known as Fermat's Little Theorem to distinguish it from the infamous Fermat's Last Theorem (which we will meet again in Book C, Chapter 12), it may not appear to be anything remarkable. Yet as the chapter unfolds we will show many applications of the theorem, and in particular we will capitalise on its immense potential in assisting in calculations involving large numbers. Deservedly it is considered to be a cornerstone in the development of number theory.

The role of Fermat's Theorem in solving certain problems will lead us into the subject of primitive roots (referred to in the extract from Bell above). We will pursue these ideas more comprehensively in Book C, Chapter 9, but here we use them to explain the periodic behaviour of the decimal expansions of fractions.

This chapter also introduces a second important result, Wilson's Theorem, which is similar in nature to Fermat's Little Theorem although it is not connected in any way. This theorem asserts that if p is prime then $(p-1)! \equiv -1 \pmod{p}$. Wilson's Theorem shares with Fermat's Little Theorem the feature that both are concerned with properties of *all* prime numbers. What is more, they have another rare feature in common: for both there is a relatively simple general proof available!

The statements of both Fermat's and Wilson's Theorems are implications, that is, they are of the form *if A holds then B holds*. For such statements it is natural to ask about the converse: *if B holds then A holds*. The converse of Wilson's Theorem, which states that for $p > 1$, if $(p-1)! \equiv -1 \pmod{p}$ then p is prime, will be seen to be true. On the other hand, the converse of Fermat's Little Theorem will be seen to be false. It is not true that if, for all integers a, we have that $a^p - a$ is divisible by p, then p must be prime.

Chapter guide

This chapter consists of five sections, the first and third of which, on Fermat's Little Theorem and Wilson's Theorem, are slightly shorter than the second and fourth. Section 2 applies Fermat's Little Theorem to explore properties of decimal representation, while Section 4 explores the notion of congruence applied to polynomials. The fifth section is a brief, non-assessed section on two combinatorial problems that, respectively, illustrate Fermat's Little Theorem and Wilson's Theorem.

1 Fermat's Little Theorem

This first section states and proves Fermat's Little Theorem, and explores why and when the converse statement does not hold.

1.1 Statement and proof of Fermat's Little Theorem

Figure 1.1 shows the values of successive powers of an integer a modulo each of the primes $p = 3, 5$ and 7, where a ranges over $1, \ldots, p-1$.

mod 3		
a	1	2
a^2	1	1

mod 5				
a	1	2	3	4
a^2	1	4	4	1
a^3	1	3	2	4
a^4	1	1	1	1

mod 7						
a	1	2	3	4	5	6
a^2	1	4	2	2	4	1
a^3	1	1	6	1	6	6
a^4	1	2	4	4	2	1
a^5	1	4	5	2	3	6
a^6	1	1	1	1	1	1

Figure 1.1 Powers modulo a prime

There are some interesting patterns beginning to emerge from these tables. Notice how some power in each column is congruent to 1, and thereafter the powers 'cycle'. For instance, if we continue to determine more and more powers of 4 modulo 7 then we get the sequence

$$4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, \ldots,$$

with the cycle 4, 2, 1 of length 3.

The powers of 5 modulo 7 give the sequence

$$5, 4, 6, 2, 3, 1, 5, 4, 6, 2, 3, 1, \ldots,$$

with a cycle 5, 4, 6, 2, 3, 1 of length 6.

In particular, notice how the last listed row of each table consists entirely of 1s. That is, for these three prime moduli, $a^{p-1} \equiv 1 \pmod{p}$ for each $a \not\equiv 0 \pmod{p}$. The result illustrated here holds generally for any prime p, and was first formulated by Fermat. This result may look slightly different from the claim in the introduction that, for all integers a, $a^p - a$ is divisible by p, but you will shortly see that they are indeed the same.

Theorem 1.1 *Fermat's Little Theorem*

If p is a prime and a is any integer with $\mathrm{hcf}(a, p) = 1$, then

$$a^{p-1} \equiv 1 \pmod{p}.$$

The condition $\mathrm{hcf}(a, p) = 1$ is equivalent to each of $a \not\equiv 0 \pmod{p}$ and 'p does not divide a'. Depending on the application we will use the most appropriate form.

Proof Consider the set of $p-1$ integers

$$\{a, 2a, 3a, \ldots, (p-1)a\}.$$

None of these numbers is congruent to 0 modulo p, for if $ra \equiv 0 \pmod{p}$, Euclid's Lemma gives $r \equiv 0 \pmod{p}$ or $a \equiv 0 \pmod{p}$, neither of which is the case here. What is more, no two of these numbers are congruent to each other modulo p, for if we had

$$ra \equiv sa \pmod{p}, \text{ where } 1 \leq r \leq p-1 \text{ and } 1 \leq s \leq p-1,$$

then cancellation of the a, which we can do since $\text{hcf}(a, p) = 1$, gives $r \equiv s \pmod{p}$ and so $r = s$. It follows that these $p-1$ numbers are congruent, in some order, to the numbers in the set

$$\{1, 2, 3, \ldots, p-1\}$$

consisting of the least positive residues of p, excluding zero.

So, by multiplying all the numbers in each set together, we get

$$a \times 2a \times 3a \times \cdots \times (p-1)a \equiv 1 \times 2 \times 3 \times \cdots \times (p-1) \pmod{p},$$

which gives

$$a^{p-1} \times (p-1)! \equiv (p-1)! \pmod{p}.$$

Since p does not divide $(p-1)!$ the latter can be cancelled throughout the congruence, leaving $a^{p-1} \equiv 1 \pmod{p}$, as required. ■

It is certainly not true that $a^{p-1} \equiv 1 \pmod{p}$ when $a \equiv 0 \pmod{p}$, for $0^{p-1} \equiv 0 \pmod{p}$. However Fermat's Little Theorem, which we will refer to also in the abbreviated form FLT, can be expressed in an alternative way that includes the case $a \equiv 0 \pmod{p}$, as follows.

Fermat's Little Theorem, an alternative formulation

If p is a prime and a is any integer,

$$a^p \equiv a \pmod{p}.$$

Exercise 1.1

Prove that the two formulations of FLT are equivalent. That is, prove that for prime p:

(a) if $a^p \equiv a \pmod{p}$ for all integers a, then $a^{p-1} \equiv 1 \pmod{p}$ for all $a \not\equiv 0 \pmod{p}$

(b) if $a^{p-1} \equiv 1 \pmod{p}$ for all $a \not\equiv 0 \pmod{p}$, then $a^p \equiv a \pmod{p}$ for all integers a.

FLT has applications in many areas, as you will see. But it really comes into its own in assisting with complex calculations, as in the following worked exercise.

Worked Exercise 1.2

(a) What is the remainder when 7^{40} is divided by 17?

(b) Show that, for any odd prime p,

$$1^p + 2^p + 3^p + \cdots + (p-1)^p \text{ is divisible by } p.$$

Solution

(a) FLT tells us that $7^{16} \equiv 1 \pmod{17}$. Therefore, breaking the power 40 down in an appropriate way,

$$7^{40} = (7^{16})^2 \times 7^8 \equiv 1^2 \times 7^8 \equiv 7^8 \pmod{17}.$$

Now $7^2 = 49 \equiv -2 \pmod{17}$, and so

$$7^{40} \equiv 7^8 \equiv (7^2)^4 \equiv (-2)^4 \equiv 16 \pmod{17},$$

and the remainder on dividing 7^{40} by 17 is 16.

(b) By FLT, $a^p \equiv a \pmod{p}$ for all integers a. Therefore

$$\begin{aligned} &1^p + 2^p + 3^p + \cdots + (p-1)^p \\ &\quad \equiv 1 + 2 + 3 + \cdots + (p-1) \pmod{p} \\ &\quad \equiv \frac{p(p-1)}{2} \pmod{p}, \quad \text{sum of an arithmetic progression} \\ &\quad \equiv 0 \pmod{p}, \quad \text{since } \frac{p-1}{2} \text{ is an integer when } p \text{ is odd.} \end{aligned}$$

So p divides $1^p + 2^p + 3^p + \cdots + (p-1)^p$ for all odd primes p.

Exercise 1.2

(a) Find the remainder when:

(i) 5^{20} is divided by 7

(ii) 37^{37} is divided by 17.

(b) Let p be a prime and $\text{hcf}(a, p) = 1$. Use FLT to verify that $x \equiv a^{p-2}b \pmod{p}$ is a solution of the linear congruence $ax \equiv b \pmod{p}$. Hence solve $5x \equiv 18 \pmod{19}$.

1.2 Pseudoprimes

The converse of the alternative formulation of FLT fails to hold. That is, if $a^n \equiv a \pmod{n}$ for every integer a, it does not follow that n is prime. But counter-examples to this converse are not so easy to find. Concentration on the case $a = 2$ has historical interest. For a very long time it was thought n had to be prime for $2^n - 2$ to be divisible by n, and this was used as a test for the primality of n. Let us first put an end to that conjecture.

Worked Exercise 1.3

Show that $2^{341} \equiv 2 \pmod{341}$.

Solution

We will show that $2^{340} \equiv 1 \pmod{341}$, from which the result follows on multiplying through by 2.

341 is not a counter-example to the converse of FLT since it is not true that $a^{341} \equiv a \pmod{341}$ for *all* integers a. For example, $3^{341} \equiv 168 \pmod{341}$.

Notice that $341 = 11 \times 31$, so our line of attack will be to determine first the values of $2^{340} \pmod{11}$ and $2^{340} \pmod{31}$. As 11 and 31 are primes, we can call on FLT; this tells us that $2^{10} \equiv 1 \pmod{11}$ and that $2^{30} \equiv 1 \pmod{31}$. Working modulo 11:

$$2^{340} = (2^{10})^{34} \equiv 1^{34} \equiv 1 \pmod{11}.$$

Working modulo 31:

$$2^{340} = (2^{30})^{11} \times (2^{10}) \equiv 1^{11} \times 2^5 \times 2^5 \equiv 1 \times 1 \times 1 \equiv 1 \pmod{31}.$$

As 11 and 31 are coprime we can appeal to Theorem 1.6 of Chapter 3 to conclude, from the above two congruences, that

$$2^{340} \equiv 1 \pmod{341}.$$

Pseudoprimes are sometimes called Poulet numbers, after the French mathematician who, in 1926, computed them all up to 5×10^7.

Composite integers n with the property that n divides $2^n - 2$ are called **pseudoprimes**. They do not have to be odd numbers; the two smallest even pseudoprimes are 161 038 and 215 326. There are infinitely many pseudoprimes, although they appear to be much sparser than the primes themselves. For example, of the numbers up to 2×10^{10}, it has been shown that 882 206 716 are primes but only 19 865 are pseudoprimes. The smallest pseudoprime, discovered in 1819, is 341. The next, discovered in 1912, furnishes us with the first counter-example to the converse of FLT.

Worked Exercise 1.4

Show that $a^{561} \equiv a \pmod{561}$ for every integer a.

Solution

First, we factorise 561 as $561 = 3 \times 11 \times 17$. We will determine the value of a^{561} modulo each of 3, 11 and 17, using FLT. For any integer a,

$$a^3 \equiv a \pmod{3},\ a^{11} \equiv a \pmod{11} \text{ and } a^{17} \equiv a \pmod{17}.$$

Working modulo 3:

$$a^{561} = (a^3)^{187} \equiv a^{187} = (a^3)^{62} \times a \equiv a^{62} \times a = a^{63} = (a^3)^{21} \equiv a^{21}$$
$$= (a^3)^7 \equiv a^7 = (a^3)^2 \times a \equiv a^2 \times a = a^3 \equiv a \pmod{3}.$$

Working modulo 11:

$$a^{561} = (a^{11})^{51} \equiv a^{51} = (a^{11})^4 \times a^7 \equiv a^4 \times a^7 = a^{11} \equiv a \pmod{11}.$$

Working modulo 17:

$$a^{561} = (a^{17})^{33} \equiv a^{33} = a^{17} \times a^{16} \equiv a \times a^{16} = a^{17} \equiv a \pmod{17}.$$

Theorem 1.6 of Chapter 3 now gives $a^{561} \equiv a \pmod{561}$ for all integers a. So 561 is a counter-example to the converse of FLT.

It was proved, in 1994, that there are in fact infinitely many numbers that provide a counter-example to the converse of FLT. Such numbers are called **Carmichael numbers**.

Having two seemingly different formulations of FLT requires us to decide which one to use in a given situation. When confronted with a result about *all* integers a, as in Worked Exercise 1.4, you must use $a^p \equiv a \pmod{p}$, to ensure that the argument caters for the case $a \equiv 0 \pmod{p}$. On the other hand, when dealing with a result concerning integers a for which $a \not\equiv 0 \pmod{p}$, as in Worked Exercise 1.3, although $a^p \equiv a \pmod{p}$ still applies, the alternative congruence $a^{p-1} \equiv 1 \pmod{p}$ invariably leads to simpler computations.

Exercise 1.3

(a) Prove that 168 divides $a^6 - 1$, where a is any integer for which $\text{hcf}(a, 42) = 1$.

(b) For any integer a, show that a^5 and a have the same units digit. Deduce that a^{100} has the same units digit as a^4.

Exercise 1.4

(a) Find the least positive residue modulo 13 of 5^{12}, 5^{14} and 1996^{14}.

(b) Use FLT to show that:

(i) $3^{50} + 5^{50}$ is divisible by 17

(ii) $2^{100} + 3^{100}$ is divisible by 97.

(c) What is the remainder when 7^{154} is divided by 155?

Hint: $155 = 5 \times 31$.

(d) Show that $a^7 \equiv a \pmod{42}$ for all integers a.

(e) If p and q are distinct primes, prove that $p^{q-1} + q^{p-1} \equiv 1 \pmod{pq}$.

(f) Show that $(p-1)2^{p-1} + 1$ and $(p-2)2^{p-2} + 1$ are each divisible by the odd prime p. Is it true that $(p-3)2^{p-3} + 1$ is necessarily divisible by the odd prime p?

2 Representation of fractions by decimals

We now use Fermat's Little Theorem to explore some properties of decimal representations.

2.1 Terminating and cycling decimals

When reciprocals of integers are written out as decimals, a variety of patterns emerge. A few reciprocals are shown in Table 2.1.

Table 2.1 Decimal representation of the reciprocals of 2 to 13

Reciprocal	Decimal representation
$\frac{1}{2}$	0.5
$\frac{1}{3}$	$0.333\ldots = 0.\langle 3\rangle$
$\frac{1}{4}$	0.25
$\frac{1}{5}$	0.2
$\frac{1}{6}$	$0.1666\ldots = 0.1\langle 6\rangle$
$\frac{1}{7}$	$0.142857142857142857\ldots = 0.\langle 142857\rangle$
$\frac{1}{8}$	0.125
$\frac{1}{9}$	$0.111\ldots = 0.\langle 1\rangle$
$\frac{1}{10}$	0.1
$\frac{1}{11}$	$0.090909\ldots = 0.\langle 09\rangle$
$\frac{1}{12}$	$0.08333\ldots = 0.08\langle 3\rangle$
$\frac{1}{13}$	$0.076923076923076923\ldots = 0.\langle 076923\rangle$

The 'angle bracket' notation is used to indicate that the string of digits in the brackets repeats indefinitely.

We observe that some of the decimals terminate while others go on forever. All the non-terminating ones have a **cycle** that is repeated indefinitely, such as the cycle $\langle 142857 \rangle$ of length 6 in $\frac{1}{7}$ and the cycle $\langle 09 \rangle$ of length 2 in $\frac{1}{11}$. In some of the decimals (for example $\frac{1}{3}$ and $\frac{1}{7}$) there is just the repeating cycle, while in others (for example $\frac{1}{6}$ and $\frac{1}{12}$) the cycle is preceded by some dissimilar digits. Can we explain these differences?

The **length of a cycle** is the number of digits forming the cycle.

To determine the decimal of $\frac{1}{7}$ by hand you might well proceed by long division, as illustrated on the left of Figure 2.1. To remind you of how long division works, at the boxed stage, 20 is divided by 7 giving a quotient of 2 and a remainder of 6. The remainder is then multiplied by 10 and the process continues by dividing 60 by 7. On the right we have presented the same set of calculations expressed in terms of the division algorithm.

Notice that the zero quotient in the first equation just gives the zero before the decimal point.

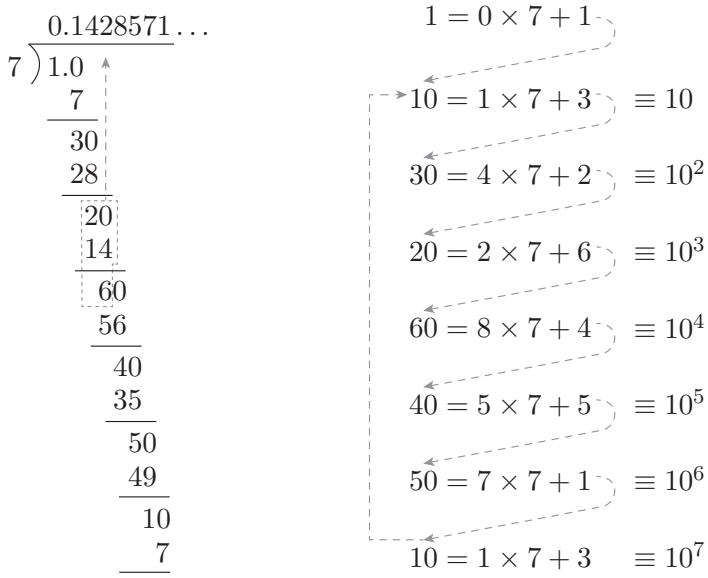


Figure 2.1 The decimal of $\frac{1}{7}$ by long division

From the system of equations we observe that the arising remainders are the least positive residues of successive powers of 10 modulo 7. The seventh equation reveals that $10^6 \equiv 1 \pmod 7$, a fact that we could have predicted by virtue of FLT. Observe also that at the seventh equation we meet the first repeated remainder, as 1 was also the remainder in the original equation. The relevance of this is that thereafter, the same sequence of equations must recur: the eighth equation is the second one again, and so on, explaining the cycling of the decimal.

There is nothing special about 7 here. For any natural number n, as there are only n residues modulo n, a remainder must repeat, and from this point onwards the sequence of equations recurs.

Exercise 2.1

Determine, by using the division algorithm, the decimal of (a) $\frac{1}{15}$ and (b) $\frac{1}{41}$.

Let us look at the general reciprocal $\frac{1}{n}$ for the integer $n > 1$. Suppose that the equations on dividing 1 by n begin

$$\begin{aligned} 1 &= 0n + 1 \\ 10 &= q_1 n + r_1 \qquad 0 \le r_1 < n \\ 10r_1 &= q_2 n + r_2 \qquad 0 \le r_2 < n \\ 10r_2 &= q_3 n + r_3 \qquad 0 \le r_3 < n \\ &\vdots \end{aligned}$$

Note that each q_i is one of the digits 0, 1, 2, ..., or 9 since $0 \le 10r_{i-1} < 10n$, and q_i is the quotient on dividing $10r_{i-1}$ by n.

The decimal of $\frac{1}{n}$ then begins $0.q_1q_2q_3\ldots$. But it is the remainders r_i that tell us when recurrence will take place. As each of the remainders r_i is one of the n integers satisfying $0 \le r_i < n$, there are at most n different equations here, and once a remainder appears for the second time subsequent equations repeat in a cycle.

Rearranging allows us to observe that $r_1 = 10 - q_1 n \equiv 10 \pmod{n}$, and so $r_2 = 10r_1 - q_2 n \equiv 10^2 \pmod{n}$. In fact we have

$$r_k \equiv 10^k \pmod{n}, \quad \text{for each } k.$$

One possibility that we have not considered is that remainder 0 arises. If this happens then the sequence of equations becomes, for some k,

$$\begin{aligned} &\vdots \\ 10r_{k-2} &= q_{k-1} n + r_{k-1} \\ 10r_{k-1} &= q_k n + 0 \\ 0 &= 0n + 0 \\ &\vdots \end{aligned}$$

This is the situation corresponding to the terminating decimal. The last non-zero equation tells us that n divides $10r_{k-1}$, from which we deduce that n divides 10^k because $r_{k-1} \equiv 10^{k-1} \pmod{n}$. Now the factors of powers of 10 are those numbers whose only prime factors are 2 and/or 5, namely the numbers of the form $2^r 5^s$, where $r \ge 0$ and $s \ge 0$. Conversely, if $n = 2^r 5^s$, then n divides 10^k where $k = \max\{r, s\}$ and the decimal of $\frac{1}{n}$ terminates after k decimal places. So the integers $1 < n < 100$ for which the decimal of $\frac{1}{n}$ terminates are $n = 2$, 4, 5, 8, 10, 16, 20, 25, 32, 40, 50, 64 and 80.

Before we move on to consider the case where the decimal does not terminate, let us record the following result.

Proposition 2.1 *Terminating decimals*

The decimal of $\frac{1}{n}$ terminates if, and only if, $n = 2^r 5^s$, for some integers $r \geq 0$ and $s \geq 0$.

2.2 The order of an integer

Let us turn to $\frac{1}{p}$ for p a prime other than 2 or 5. As $\text{hcf}(p, 10) = 1$, FLT tells us that $10^{p-1} \equiv 1 \pmod{p}$. The implication of this for the system of equations for $\frac{1}{p}$ is that the remainder in the pth equation is 1, and this will be a repeat of the remainder in the initial equation.

$$\begin{aligned} 1 &= 0 \times p + 1 \\ &\vdots \\ 10^{p-1} \equiv 1 &= 0 \times p + 1 \end{aligned}$$

So the decimal of $\frac{1}{p}$ has a cycle that begins immediately after the decimal point and that has length at most $p - 1$. We say that the length is *at most* $p - 1$ because, although we know that $10^{p-1} \equiv 1 \pmod{p}$, it might also be the case that $10^k \equiv 1 \pmod{p}$ for some smaller natural number k, and the remainder 1 would be repeated in an earlier equation. Indeed, we have already seen that this can happen. When $p = 11$ the decimal of $\frac{1}{11}$ is $0.\langle 09 \rangle$. As predicted, this has a cycle beginning at the first decimal place. But the length of this cycle is 2, rather than the suggested 10, because it is the case that $10^2 \equiv 1 \pmod{11}$. Similarly, in Exercise 2.1 you discovered that $\frac{1}{41}$ has a cycle of length 5. While FLT assured us that the cycle length could not exceed 40, since $10^{40} \equiv 1 \pmod{41}$, it is readily checked that $10^5 \equiv 1 \pmod{41}$.

We should really say that *the decimal of* $\frac{1}{41}$ has a cycle of length 5. However, when the meaning is clear we will often omit 'the decimal of' for brevity.

All this suggests that we should look with interest at increasing positive powers of 10, searching for the first one that is congruent to 1 modulo p because this least exponent will give the length of the cycle in $\frac{1}{p}$. We refer to this least exponent as the *order* of 10 modulo p. The general definition is as follows.

Definition 2.2 *The order of an integer*

If p is prime and $\text{hcf}(a, p) = 1$, then the *order* of a modulo p is the least positive integer c such that $a^c \equiv 1 \pmod{p}$.

If $\text{hcf}(a, p) \neq 1$ then $a \equiv 0 \pmod{p}$ and so $a^c \equiv 0 \pmod{p}$ for all positive integers c. In this case there is no concept of order.

This definition assumes that for $\text{hcf}(a, p) = 1$ there is a unique least positive integer c such that $a^c \equiv 1 \pmod{p}$. This is guaranteed by FLT since $a^{p-1} \equiv 1 \pmod{p}$ gives $c \leq p - 1$.

With this terminology at our disposal, we can state formally what we have discovered about the cycle in $\frac{1}{p}$.

Proposition 2.3 *The cycle length of $\frac{1}{p}$*

The length of the cycle in the decimal of $\frac{1}{p}$ is equal to the order of 10 modulo p, where p is a prime other than 2 or 5.

Exercise 2.2

Determine the order of 10 modulo 17 and hence the length of the cycle of $\frac{1}{17}$.

Table 2.2 shows the order of 10 modulo each of the primes $p < 100$ (other than 2 and 5). Notice that for nine of these primes, the order of 10 modulo p is $p - 1$. When this occurs we say that 10 is a **primitive root** of p, and in this case the successive powers of 10 run through all the $p - 1$ non-zero residues of p in some order, as illustrated for the prime 17 in Exercise 2.2.

We deal with primitive roots in general in Book C, Chapter 9. For our present purposes we are concerned only with cases where 10 is a primitive root of p.

Table 2.2 The order of 10 modulo p for primes $p < 100$, other than 2 or 5

Prime p	Order of 10 modulo p	Prime p	Order of 10 modulo p
3	1	47	46
7	6	53	13
11	2	59	58
13	6	61	60
17	16	67	33
19	18	71	35
23	22	73	8
29	28	79	13
31	15	83	41
37	3	89	44
41	5	97	96
43	21		

This is the start of a much more extensive table that illustrates how, over the years, mathematicians have devoted large portions of their lives to calculations that can only be their own reward. In 1873 William Shanks (1812–1882) published an extension of this table covering all primes up to 30 000, and he subsequently progressed the table to all primes less than 120 000. To get some idea of how much calculation this involved – and remember that we are talking about the days long before electronic

calculating devices had landed on our desks – consider the prime 61. We know that $10^{60} \equiv 1 \pmod{61}$ and we need to confirm that there is no smaller natural number k for which $10^k \equiv 1 \pmod{61}$. In Exercise 2.2 we attacked a similar problem for the prime 17 by working out all the smaller powers of 10. However, there was no need to work them all out, as the following illustrates.

Exercise 2.3

Prove that if the order of 10 modulo 61 is c, then c is a factor of 60.

By virtue of Exercise 2.3, the order of 10 modulo 61 can be determined by finding the least positive residues, modulo 61, of the following powers of 10:

$$10, 10^2, 10^3, 10^4, 10^5, 10^6, 10^{10}, 10^{12}, 10^{15}, 10^{20} \text{ and } 10^{30}.$$

The required order is the exponent in the first of these that is congruent to 1 modulo 61. If, as turns out to be the case, none of them is congruent to 1 then the order of 10 is 60. This is a considerable improvement on having to determine all the first 60 powers of 10, but it still represents a considerable amount of computation; and all this is for just one relatively small prime. No doubt Shanks had a number of shortcuts available, but to achieve what he did by hand was no mean feat.

The result of Exercise 2.3 can be generalised, for there is nothing special about the prime 61. The proof of the following result, which we will use in Book C, Chapter 9, simply follows the argument given in the solution of Exercise 2.3.

Proposition 2.4

If a has order c modulo p, where $p \geq 3$ is prime, and $a^k \equiv 1 \pmod{p}$ then c is a factor of k.

In particular, c divides $p - 1$.

Proof By the Division Algorithm, we can write

$$k = qc + r, \quad \text{where } 0 \leq r < c.$$

Now

$$a^k = a^{qc+r} = (a^c)^q \times a^r \equiv 1^q \times a^r \equiv a^r \pmod{p}$$

and as $a^k \equiv 1 \pmod{p}$, $a^r \equiv 1 \pmod{p}$. Therefore $r = 0$, for otherwise the order of a would be less than c. Hence c divides k.

As FLT gives $a^{p-1} \equiv 1 \pmod{p}$, the order c divides $p - 1$. ■

2.3 Decimal representations of rational numbers

To determine the decimal of $\frac{2}{7}$ we proceed just as for $\frac{1}{7}$, but this time the first equation involves dividing 2 by 7.

$$\begin{aligned} 2 &= 0 \times 7 + 2 \\ 20 &= 2 \times 7 + 6 \\ 60 &= 8 \times 7 + 4 \\ 40 &= 5 \times 7 + 5 \\ 50 &= 7 \times 7 + 1 \\ 10 &= 1 \times 7 + 3 \\ 30 &= 4 \times 7 + 2 \end{aligned}$$

Notice that we get the same cycle of six equations as we had for $\frac{1}{7}$ but starting with a different equation. So the same six digits, in the same cyclic order but this time starting with the digit 2, form the cycle $\langle 285714 \rangle$ of $\frac{2}{7}$. In the same way you could check that $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$ and $\frac{6}{7}$ have decimals consisting of the same cycle of digits as that of $\frac{1}{7}$, but with appropriate starting points as shown below.

$$\begin{aligned} \tfrac{1}{7} &= 0.\langle 142857 \rangle \\ \tfrac{2}{7} &= 0.\langle 285714 \rangle \\ \tfrac{3}{7} &= 0.\langle 428571 \rangle \\ \tfrac{4}{7} &= 0.\langle 571428 \rangle \\ \tfrac{5}{7} &= 0.\langle 714285 \rangle \\ \tfrac{6}{7} &= 0.\langle 857142 \rangle \end{aligned}$$

Although the proof of this result is not too demanding, we have opted not to include it here.

This situation is similar for any prime p having 10 as a primitive root. For such a prime the cycle of $\frac{1}{p}$ has length $p-1$ and each $\frac{r}{p}$, for $r = 1, 2, \ldots, p-1$, has the same cycle of $p-1$ digits but each with its own starting position.

Exercise 2.4

Given that $\frac{1}{17} = 0.\langle 0588235294117647 \rangle$, determine the decimals of $\frac{2}{17}$ and $\frac{9}{17}$.

We would be lacking in curiosity if we did not now ask whether or not any of this 'recycling' occurs when 10 is not a primitive root of prime p. Look at $p = 13$. In finding the decimal of $\frac{1}{13}$ we get the following set of equations.

$$\begin{aligned} 1 &= 0 \times 13 + 1 \\ 10 &= 0 \times 13 + 10 \\ 100 &= 7 \times 13 + 9 \\ 90 &= 6 \times 13 + 12 \\ 120 &= 9 \times 13 + 3 \\ 30 &= 2 \times 13 + 4 \\ 40 &= 3 \times 13 + 1 \end{aligned}$$

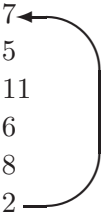
As the cycle has length 6, the order of 10 modulo 13 is 6 and so 10 is not a primitive root of 13. By taking each of its six starting points, these equations give the following six decimals:

$$\tfrac{1}{13} = 0.\langle 076923\rangle,\ \tfrac{3}{13} = 0.\langle 230769\rangle,\ \tfrac{4}{13} = 0.\langle 307692\rangle,$$
$$\tfrac{9}{13} = 0.\langle 692307\rangle,\ \tfrac{10}{13} = 0.\langle 769230\rangle,\ \tfrac{12}{13} = 0.\langle 923076\rangle.$$

The numerator in each fraction is the remainder in the equation that precedes the start of the cycle.

In finding the decimal of $\frac{2}{13}$ we get the following set of equations.

$$\begin{aligned} 2 &= 0 \times 13 + 2\\ 20 &= 1 \times 13 + 7\\ 70 &= 5 \times 13 + 5\\ 50 &= 3 \times 13 + 11\\ 110 &= 8 \times 13 + 6\\ 60 &= 4 \times 13 + 8\\ 80 &= 6 \times 13 + 2 \end{aligned}$$



These equations give a second set of six decimals:

$$\tfrac{2}{13} = 0.\langle 153846\rangle,\ \tfrac{5}{13} = 0.\langle 384615\rangle,\ \tfrac{6}{13} = 0.\langle 461538\rangle,$$
$$\tfrac{7}{13} = 0.\langle 538461\rangle,\ \tfrac{8}{13} = 0.\langle 615384\rangle,\ \tfrac{11}{13} = 0.\langle 846153\rangle.$$

There is nothing exceptional in this behaviour for the prime 13: it is typical of what happens for any prime p. Suppose that the order of 10 modulo p is c. It turns out that the decimals of $\frac{r}{p}$, for $1 \le r < p$ form sets, each set consisting of c members with the same cycle of c digits varying only in their starting points. For example, consider $p = 41$. The order of 10 modulo 41 is 5 (see the solution of Exercise 2.1) and the 40 decimals $\frac{r}{41}$ have cycles of five digits. So these decimals will form eight sets, each set consisting of the same cycle but with the five different starting points. These are displayed in Table 2.3.

Table 2.3 The sets of cycles of $\frac{r}{41}$, for $1 \le r < 41$

Set	Decimals				
1	$\frac{1}{41} = 0.\langle 02439\rangle$	$\frac{10}{41} = 0.\langle 24390\rangle$	$\frac{16}{41} = 0.\langle 39024\rangle$	$\frac{18}{41} = 0.\langle 43902\rangle$	$\frac{37}{41} = 0.\langle 90243\rangle$
2	$\frac{2}{41} = 0.\langle 04878\rangle$	$\frac{20}{41} = 0.\langle 48780\rangle$	$\frac{32}{41} = 0.\langle 78048\rangle$	$\frac{33}{41} = 0.\langle 80487\rangle$	$\frac{36}{41} = 0.\langle 87804\rangle$
3	$\frac{3}{41} = 0.\langle 07317\rangle$	$\frac{7}{41} = 0.\langle 17073\rangle$	$\frac{13}{41} = 0.\langle 31707\rangle$	$\frac{29}{41} = 0.\langle 70731\rangle$	$\frac{30}{41} = 0.\langle 73170\rangle$
4	$\frac{4}{41} = 0.\langle 09756\rangle$	$\frac{23}{41} = 0.\langle 56097\rangle$	$\frac{25}{41} = 0.\langle 60975\rangle$	$\frac{31}{41} = 0.\langle 75609\rangle$	$\frac{40}{41} = 0.\langle 97560\rangle$
5	$\frac{5}{41} = 0.\langle 12195\rangle$	$\frac{8}{41} = 0.\langle 19512\rangle$	$\frac{9}{41} = 0.\langle 21951\rangle$	$\frac{21}{41} = 0.\langle 51219\rangle$	$\frac{39}{41} = 0.\langle 95121\rangle$
6	$\frac{6}{41} = 0.\langle 14634\rangle$	$\frac{14}{41} = 0.\langle 34146\rangle$	$\frac{17}{41} = 0.\langle 41463\rangle$	$\frac{19}{41} = 0.\langle 46341\rangle$	$\frac{26}{51} = 0.\langle 63414\rangle$
7	$\frac{11}{41} = 0.\langle 26829\rangle$	$\frac{12}{41} = 0.\langle 29268\rangle$	$\frac{28}{41} = 0.\langle 68292\rangle$	$\frac{34}{41} = 0.\langle 82926\rangle$	$\frac{38}{41} = 0.\langle 92682\rangle$
8	$\frac{15}{41} = 0.\langle 36585\rangle$	$\frac{22}{41} = 0.\langle 53658\rangle$	$\frac{24}{41} = 0.\langle 58536\rangle$	$\frac{27}{41} = 0.\langle 65853\rangle$	$\frac{35}{41} = 0.\langle 85365\rangle$

We leave decimals at this point, aware that there are further areas for investigation. In particular, we have not pursued reciprocals of composite numbers, other than to recognise which ones terminate. As a parting

observation, we mention that the decimal of $\frac{1}{49}$ is a cycle of length 42:

$$\tfrac{1}{49} = 0.\langle 020408163265306122448979591836734693877551\rangle,$$

and that the 42 decimals obtained from this cycle by letting each digit of the cycle lead are the proper fractions $\frac{k}{49}$, where $1 \leq k < 49$; that is, those with $\text{hcf}(k, 49) = 1$. For example,

$$\tfrac{11}{49} = 0.\langle 224489795918367346938775510204081632653061\rangle.$$

However, of all composite integers less than 100, 49 is the only one that exhibits this elegant behaviour of its proper fractions. The relevance of the cycle length of $\frac{1}{49}$ will become apparent once you have met Euler's generalisation of FLT, in Book C, Chapter 9.

A **proper fraction** is one of the form $\frac{m}{n}$, where $\text{hcf}(m, n) = 1$; that is, a fraction in which all common factors of the numerator and denominator have been cancelled.

Exercise 2.5

(a) Leibniz claimed, in 1677, that for any natural number n with $\text{hcf}(n, 10) = 1$, the length of the cycle of $\frac{1}{n}$ is a factor of $n - 1$. Determine the decimal fraction for $\frac{1}{21}$ and prove him wrong.

(b) From Proposition 2.3 it can be deduced that for each natural number n, there are at most a finite number of primes p for which the length of the cycle of $\frac{1}{p}$ is n. Explain how.

(c) By first determining the prime decomposition of $10^4 - 1$, find all the primes p for which $\frac{1}{p}$ has a cycle of length 4. From the decomposition of $10^6 - 1$, find all primes p for which $\frac{1}{p}$ consists of a cycle of length 6.

(d) Consider an ordinary pack of 52 playing cards. In the **perfect faro shuffle** the pack is cut into two equal halves, which are then placed (flicked) alternately, as illustrated in Figure 2.2.

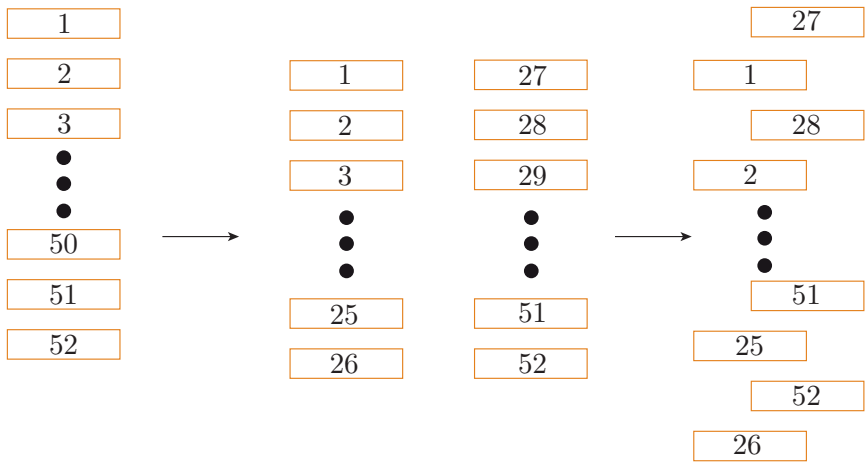


Figure 2.2 The perfect faro shuffle

The cards that initially occupy positions 1, 2, ..., 26 are moved to positions 2, 4, ..., 52 whilst the cards initially in positions 27, 28, ..., 52 are moved to positions 1, 3, ..., 51. Thus the card starting at position x moves to position y, where $1 \leq y \leq 52$ and

$y \equiv 2x \pmod{53}$. After n such shuffles this card will occupy position $2^n x \pmod{53}$.

(i) After how many perfect faro shuffles will the pack return to its initial order for the first time?

(ii) If two jokers are included in the pack, making 54 cards in all, after how many perfect faro shuffles would this pack be restored to the original order for the first time?

3 Wilson's Theorem

Let us turn now to a result that is not of very much practical use but one that, nevertheless, has a prominent place in the development of number theory. The English mathematician Edward Waring (*c.*1736–1798) reported, in his book *Meditationes Algebraicae* (1770), a result claimed by one of his former students, John Wilson (1741–1793). Wilson had asserted that for each prime p, the number

$$\frac{(p-1)!+1}{p}$$

is an integer. Wilson appears to have guessed the result on empirical evidence, for neither he nor Waring could supply a proof. But shortly afterwards, the French mathematician Lagrange (1736–1813) gave a proof of the result. Throughout mathematical literature the result has become known as Wilson's Theorem, although a manuscript written in 1682 indicates that Leibniz was aware of the result, and how to prove it, long before Wilson came on the scene.

The language of congruence gives us a neater way of expressing the result.

Theorem 3.1 *Wilson's Theorem*

If p is prime then $(p-1)! \equiv -1 \pmod{p}$.

First we look at the particular case of the prime 17 to illustrate the idea behind the proof that we are going to give. For $p = 17$ the theorem claims that $16! \equiv -1 \pmod{17}$.

To evaluate 16! we could multiply together the sixteen numbers from 1 to 16 inclusive, reducing the answer modulo 17. But notice how the numbers from 2 to 15 inclusive pair up so that their product is congruent to 1 modulo 17.

$$\begin{aligned} 2 \times 9 &\equiv 1 \pmod{17} & 3 \times 6 &\equiv 1 \pmod{17} \\ 4 \times 13 &\equiv 1 \pmod{17} & 5 \times 7 &\equiv 1 \pmod{17} \\ 8 \times 15 &\equiv 1 \pmod{17} & 10 \times 12 &\equiv 1 \pmod{17} \\ 11 \times 14 &\equiv 1 \pmod{17} \end{aligned}$$

This observation gives us that

$$15! = 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11 \times 12 \times 13 \times 14 \times 15$$
$$\equiv 1^7 \equiv 1 \pmod{17},$$

and so

$$16! = 15! \times 16 \equiv 1 \times 16 = 16 \equiv -1 \pmod{17}.$$

The substance of our general proof lies in showing that this convenient pairing of factors in $(p-2)!$ always occurs. This particular proof was first given by Gauss.

Proof of Wilson's Theorem (Theorem 3.1) The observations $(2-1)! = 1 \equiv -1 \pmod 2$ and $(3-1)! = 2 \equiv -1 \pmod 3$ confirm the result for $p = 2$ and $p = 3$, so we can confine attention to $p \geq 5$.

Consider the set of $p-3$ integers $S = \{2, 3, 4, \ldots, p-2\}$. If $a \in S$ then, as $\text{hcf}(a, p) = 1$, the linear congruence $ax \equiv 1 \pmod p$ has, by Theorem 3.4 of Chapter 3, a *unique* least positive solution modulo p. Let the solution be a', so that $aa' \equiv 1 \pmod p$. Now $a' \neq 1$ because $a' = 1$ would give $a \equiv 1 \pmod p$, contradicting $a \in S$. Similarly $a' \neq p-1$ because $a' = p-1$ would imply $a \equiv -1 \pmod p$, again contradicting $a \in S$. Hence $a' \in S$.

Finally $a \neq a'$, for otherwise $aa' = a^2 \equiv 1 \pmod p$, which would give that p divides $a^2 - 1$. But then p divides $(a-1)(a+1)$, whereupon Euclid's Lemma gives p divides $a-1$ or p divides $a+1$, which amount to $a = 1$ or $a = p-1$ respectively, each of which contradicts $a \in S$.

Note that $p-3$ is even as p is odd.

We conclude that for each element $a \in S$ there is a unique element $a' \in S$, $a \neq a'$, for which $aa' \equiv 1 \pmod p$. Thus the elements of S form $\frac{p-3}{2}$ distinct pairs a and a' with $aa' \equiv 1 \pmod p$. Multiplying these congruences together, each element of S is involved exactly once, and so

$$2 \times 3 \times 4 \times \cdots \times (p-2) \equiv 1 \pmod p.$$

Multiplying both sides of this congruence by $(p-1)$ gives

$$2 \times 3 \times 4 \times \cdots \times (p-2) \times (p-1) \equiv p-1 \pmod p,$$

that is,

$$(p-1)! \equiv -1 \pmod p,$$

completing the proof. ■

The converse of Wilson's Theorem is also true, and its proof is relatively simple.

Proposition 3.2 *Converse of Wilson's Theorem*

If $n > 1$ is an integer and $(n-1)! \equiv -1 \pmod n$ then n is prime.

Proof Suppose that n divides $(n-1)!+1$. Let r be a positive factor of n, where $r < n$. Then as

$$(n-1)! = 1 \times 2 \times 3 \times \cdots \times r \times \cdots \times (n-1),$$

r divides $(n-1)!$. But r divides $(n-1)!+1$, and so r divides 1, from which we conclude that $r = 1$. That is, n has no positive factors other than itself and 1, and therefore is prime. ■

Exercise 3.1

Find the smallest prime factor of:

(a) $18!+1$

(b) $29!-1$.

Wilson's Theorem and its converse together give a characterisation of prime numbers: $n > 1$ is prime if, and only if, n divides $(n-1)!+1$. Unfortunately it is not of much practical use because of the amount of computation that would be involved in using it. For instance, to prove that 43 is prime by this method involves checking whether or not $42!+1$ is divisible by 43. Even working modulo 43, the evaluation of $42!$ is quite demanding, and that is for a relatively small prime. However, Wilson's Theorem can be put to work in other ways. In the next worked exercise it is used to discover a solution of a *quadratic congruence*. The theory of quadratic congruences has great importance in number theory and we devote a substantial part of Book C, Chapter 10 to its study.

Worked Exercise 3.3

Show that the congruence $x^2+1 \equiv 0 \pmod{p}$, where p is an odd prime, has a solution if, and only if, $p \equiv 1 \pmod{4}$.

Solution

We first use Wilson's Theorem to show that the congruence has a solution when $p = 4k+1$. In this case $(p-1)! = (4k)! \equiv -1 \pmod{4k+1}$. That is,

$$\begin{aligned}(p-1)! &= 1 \times 2 \times 3 \times \cdots \times (2k) \times (2k+1) \times \cdots \times (4k-1) \times (4k) \\ &\equiv -1 \pmod{4k+1}.\end{aligned}$$

Now, working modulo $4k+1$ we have

$$\begin{aligned}4k &\equiv -1 \\ 4k-1 &\equiv -2 \\ &\vdots \\ 2k+1 &\equiv -2k,\end{aligned}$$

and substituting these values into the expression for $(p-1)!$ we obtain

$$\begin{aligned}-1 &\equiv (p-1)! \pmod{4k+1}\\ &\equiv 1\times 2\times\cdots\times(2k)\times(-2k)\times\cdots\times(-2)\times(-1)\pmod{4k+1}\\ &\equiv 1\times 2\times\cdots\times(2k)\times(-1)^{2k}\times(2k)\times\cdots\times 2\times 1\pmod{4k+1}\\ &\equiv 1^2\times 2^2\times\cdots\times(2k)^2\pmod{4k+1}\\ &\equiv ((2k)!)^2\pmod{4k+1}.\end{aligned}$$

Hence $x=(2k)!$ satisfies $x^2+1\equiv 0\pmod{4k+1}$ and so the congruence has a solution.

We now show that if $p\not\equiv 1\pmod 4$ then there is no solution. In this case the odd prime is of the form $4k+3$. Heading for a contradiction, suppose that the congruence $x^2+1\equiv 0\pmod{4k+3}$ has a solution; that is, there is an integer a such that $a^2\equiv -1\pmod{4k+3}$. Then, applying FLT for prime $4k+3$,

$$a^{4k+2}\equiv 1\pmod{4k+3}.$$

But

$$a^{4k+2}=(a^2)^{2k+1}\equiv(-1)^{2k+1}\equiv -1\pmod{4k+3},$$

giving the contradiction $1\equiv -1\pmod{4k+3}$.

Exercise 3.2

(a) Find two solutions of the quadratic congruence $x^2+1\equiv 0\pmod{29}$.

(b) Prove that, for each odd prime p,

$$\begin{aligned}1^2\times 3^2\times\cdots\times(p-2)^2 &\equiv 2^2\times 4^2\times\cdots\times(p-1)^2\\ &\equiv(-1)^{(p+1)/2}\pmod p.\end{aligned}$$

Hint: use Wilson's Theorem together with the fact that $2k\equiv -(p-2k)\pmod p$.

As a final deduction from Wilson's Theorem, we observe that it tells us that there are infinitely many integers n for which $n!+1$ is composite: simply choose $n=p-1$ for any prime $p>3$.

To see this, note that p is known to be a factor of $(p-1)!+1$, implying that the latter is composite except when $(p-1)!+1=p$. This condition occurs only for $p=2$ and $p=3$.

For other values of n the number $n!+1$ may be either prime or composite, and it remains an unanswered question as to whether or not there are infinitely many primes of the form $n!+1$. For values of n up to 100, those for which $n!+1$ is prime are $n=1$, 2, 3, 11, 27, 37, 41, 73 and 77.

Exercise 3.3

(a) If, for some a, the numbers a, a^2, $a^3, \ldots, a^{p-2}$ are congruent modulo p to the numbers 2, 3, 4, ..., $p-1$, in some order, then it follows that the order of a modulo p is $p-1$ (that is, a is a primitive root of p). In particular, observe that

$$a^k a^{p-k-1} = a^{p-1} \equiv 1 \,(\text{mod } p).$$

Show that the order of 2 modulo 29 is 28 and use the given observation to find the distribution of $\{2, 3, \ldots, 27\}$ into 13 pairs in such a way that the product of each pair is congruent to 1 modulo 29. Hence illustrate the proof of Wilson's Theorem for the prime 29.

(b) If p is a prime and a is any integer, prove that p divides both

$$a^p + a \times (p-1)! \quad \text{and} \quad a + a^p \times (p-1)!$$

(c) For p prime, prove that

$$(p-1)! \equiv p-1 \,(\text{mod } 1+2+3+\cdots+(p-1)).$$

4 Polynomial congruences

In Chapter 3 we saw how to solve linear congruences. Here, with the help of Fermat's Little Theorem and Wilson's Theorem, we will explore solutions for polynomial congruences of higher degree.

4.1 Lagrange's Theorem

In preparation for the investigation of linear congruences in Chapter 3, we briefly introduced the notion of polynomial congruences and their solutions. For convenience we reiterate the definitions here.

Definition 4.1 *Polynomial congruences and their solutions*

A polynomial congruence is an expression

$$P(x) = c_r x^r + c_{r-1} x^{r-1} + \cdots + c_1 x + c_0 \equiv 0 \,(\text{mod } n),$$

where $P(x)$ is a polynomial of degree $r \geq 0$ with integer coefficients.

An integer a is a solution of the polynomial congruence $P(x) \equiv 0 \,(\text{mod } n)$ if, and only if, $P(a) \equiv 0 \,(\text{mod } n)$.

The number of solutions of a polynomial congruence is the number of incongruent solutions modulo n.

In the congruence context we are concerned exclusively with integers, and consequently we are forced to restrict attention to polynomials with integer coefficients. Hence, in this section, when we mention a polynomial it will be assumed that it has integer coefficients.

It follows from these definitions that the polynomial congruence $P(x) \equiv 0 \pmod{n}$ has at most n solutions, and these could be discovered by finding which integers in a complete set of residues, such as $\{0, 1, 2, \ldots, n-1\}$, are solutions.

Exercise 4.1

Solve the following polynomial congruences.

(a) $x^3 - x \equiv 0 \pmod{6}$

(b) $x^3 - x \equiv 0 \pmod{8}$

(c) $x^2 + x + 1 \equiv 0 \pmod{7}$

(d) $2x^2 + 9x + 15 \equiv 0 \pmod{7}$

Now consider the polynomial congruence

$$x^{20} + 3x^{14} + 8x^{10} + 3x^2 + 6 \equiv 0 \pmod{7}.$$

This still may not appear too daunting. After all, we could solve the congruence by exhaustion, evaluating the left-hand side for $x = 0$, 1, 2, 3, 4, 5 and 6. But even that involves a good deal of computation, and if we change the modulus to something significantly larger this approach would be impractical. What we need is a more sophisticated, systematic approach, and here FLT comes to our aid.

FLT informs us that, for all integers x, $x^7 \equiv x \pmod{7}$, and so

$$x^{20} = (x^7)^2 x^6 \equiv x^2 x^6 = x^7 x \equiv x^2 \pmod{7}.$$

Similarly, $x^{14} \equiv x^2 \pmod{7}$ and $x^{10} \equiv x^4 \pmod{7}$. Therefore solving the original congruence is equivalent to solving

$$x^2 + 3x^2 + 8x^4 + 3x^2 + 6 = 8x^4 + 7x^2 + 6 \equiv 0 \pmod{7}.$$

Simplifying further by reducing the coefficients, $8 \equiv 1 \pmod{7}$, etc., we arrive at the equivalent congruence

$$x^4 - 1 \equiv 0 \pmod{7}.$$

Checking the fourth powers of 0, ± 1, ± 2 and ± 3, we find that the congruence has just the two solutions $x \equiv \pm 1 \pmod{7}$.

This example illustrates a useful role for FLT in solving polynomial congruences, offering the possibility of replacing the polynomial in question with one of smaller degree. We are not going to develop any general theory for solving such congruences here but, rather, just touch the beginnings of the subject through examples. What we will find is that when working to a modulus that is not prime, unexpected and unwanted things can happen; but when the modulus is prime the situation is more clear-cut. Our

principal goal is a theorem of Lagrange that gives an insight into the number of solutions to a polynomial congruence with prime modulus.

You can read more about the life of Lagrange in the *History Reader* for this topic.

One well-known principle from the theory of equations, the so-called Fundamental Theorem of Algebra, asserts that a polynomial equation of degree n with real coefficients has at the most n roots (solutions). We already have evidence that the same principle cannot hold for polynomial congruences; for example, we witnessed in Exercise 4.1 a polynomial of degree 3 for which the associated congruence modulo 6 has more than three solutions. Indeed, we have seen that linear congruences (degree 1) do not necessarily have unique solutions; the congruence $2x \equiv 2 \pmod 4$ has the two solutions $x \equiv 1, 3 \pmod 4$. However, looking back through the examples at hand, wherever the modulus is a prime this 'misbehaviour' does not occur. Lagrange, in 1768, proved that the number of solutions of a polynomial congruence in which the modulus is prime is less than or equal to its degree.

But first we must clarify what we mean by the degree of a polynomial congruence. Consider the following congruence:

$$7x^3 + 4x^2 - 3 \equiv 0 \pmod 7.$$

The polynomial involved has degree 3. However, when we turn to congruence modulo 7, the leading term, $7x^3$, vanishes since $7 \equiv 0 \pmod 7$. Hence, modulo 7, this polynomial has degree 2 because the leading term with non-zero coefficient modulo 7 is $4x^2$.

By the *leading term* in a polynomial $P(x)$ we mean the (non-zero) term involving the highest power of x. By 'non-zero modulo n' we mean 'not congruent to zero modulo n'.

Definition 4.2 *Degree of a polynomial congruence*

If $P(x) = c_r x^r + c_{r-1}x^{r-1} + \cdots + c_1 x + c_0$ then the polynomial $P(x)$ has degree k modulo n if, after removing each term in which the coefficient $c_i \equiv 0 \pmod n$, the leading term remaining is $c_k x^k$.

If every coefficient $c_i \equiv 0 \pmod n$ then the polynomial is not assigned a degree modulo n.

The degree of the polynomial congruence $P(x) \equiv 0 \pmod n$ is the degree of $P(x)$ modulo n.

Exercise 4.2

What are the degrees of the following polynomial congruences?

(a) $3x^6 + 7x^2 - 2 \equiv 0 \pmod 5$

(b) $10x^3 + 3x + 1 \equiv 0 \pmod 5$

(c) $15x^2 - 10x + 2 \equiv 0 \pmod 5$

We are now ready to state our main result of this section. Its proof will come in the next subsection.

Theorem 4.3 *Lagrange's Theorem*

Let p be a prime. A polynomial congruence $P(x) \equiv 0 \pmod{p}$ of degree $k \geq 1$ can have at most k solutions.

4.2 Factorising polynomial congruences

The proof that we give for Lagrange's Theorem follows closely one of the methods of proving the Fundamental Theorem of Algebra in which the polynomial $P(x)$ is effectively factorised. To this end we will first prove a subsidiary result that introduces the idea of factors in polynomial congruences.

Lemma 4.4

Let b be a solution of the polynomial congruence $P(x) \equiv 0 \pmod{p}$ of degree $k \geq 1$. Then $P(x) \equiv (x-b)P_1(x) \pmod{p}$, where $P_1(x)$ is a polynomial of degree $k-1$ modulo p.

Proof Suppose that $P(x) = c_k x^k + c_{k-1}x^{k-1} + \cdots + c_1 x + c_0$, where $c_k \not\equiv 0 \pmod{p}$. Since $P(b) \equiv 0 \pmod{p}$, we have $P(x) \equiv P(x) - P(b) \pmod{p}$. Now

$$P(x) - P(b) = c_k(x^k - b^k) + c_{k-1}(x^{k-1} - b^{k-1}) + \cdots + c_1(x-b).$$

Each term in the sum on the right-hand side has a factor $(x-b)$ since

$$x^r - b^r = (x-b)(x^{r-1} + bx^{r-2} + b^2x^{r-3} + \cdots + b^{r-2}x + b^{r-1}).$$

Hence

$$P(x) \equiv P(x) - P(b) \equiv (x-b)P_1(x) \pmod{p},$$

where the polynomial $P_1(x)$ has integer coefficients and leading term $c_k x^{k-1}$. As $c_k \not\equiv 0 \pmod{p}$, $P_1(x)$ has degree $k-1$ modulo p. ■

In the statement of Lemma 4.4 we called the modulus p, but we did not stipulate that p was prime. Indeed, the above proof holds true for p composite. However, when we progress from a single factor to our next result, which pulls out a factor for each solution of the congruence, we do need the modulus to be prime.

Proposition 4.5 *Factorising a polynomial modulo p*

Let p be prime and let $b_1, b_2, \ldots, b_r$ be incongruent solutions of the polynomial congruence $P(x) \equiv 0 \pmod{p}$ of degree $k \geq r$. Then

$$P(x) \equiv (x - b_1)(x - b_2) \cdots (x - b_r)P_r(x) \pmod{p},$$

where the polynomial $P_r(x)$ has degree $k - r$ modulo p.

Proof Lemma 4.4 tells us that $P(x) \equiv (x - b_1)P_1(x) \pmod{p}$, where $P_1(x)$ has degree $k - 1$ modulo p. Now $P(b_2) \equiv 0 \pmod{p}$, and so

$$(b_2 - b_1)P_1(b_2) \equiv 0 \pmod{p}.$$

But p does not divide $b_2 - b_1$, since b_1 and b_2 are incongruent solutions modulo p. Hence $P_1(b_2) \equiv 0 \pmod{p}$ by Euclid's Lemma. This allows us to invoke Lemma 4.4 again:

This is where we require the modulus p to be prime.

$$P_1(x) \equiv (x - b_2)P_2(x) \pmod{p},$$

where $P_2(x)$ has degree $k - 2$ modulo p.

Substituting for $P_1(x)$ gives

$$P(x) \equiv (x - b_1)(x - b_2)P_2(x) \pmod{p}.$$

Continuing, from $P(b_3) \equiv 0 \pmod{p}$ we deduce that $P_2(b_3) \equiv 0 \pmod{p}$ and hence that

$$P_2(x) \equiv (x - b_3)P_3(x) \pmod{p},$$

where $P_3(x)$ has degree $k - 3$ modulo p.

The desired result is reached when all r incongruent solutions have been processed in this way. ■

Let us look at a worked exercise to illustrate the proof of Proposition 4.5.

Worked Exercise 4.6

Let $P(x) = x^4 - x^3 + 5x^2 - 3x - 2$. Confirm that $x \equiv 1, 3 \pmod{11}$ are two solutions of $P(x) \equiv 0 \pmod{11}$ and find $P_2(x)$ such that $P(x) \equiv (x - 1)(x - 3)P_2(x) \pmod{11}$.

Solution

$$P(1) = 1 - 1 + 5 - 3 - 2 \equiv 0 \pmod{11}$$

and

$$P(3) = 81 - 27 + 45 - 9 - 2 = 88 \equiv 0 \pmod{11},$$

so both are solutions as claimed.

Writing

$$x^4 - x^3 + 5x^2 - 3x - 2 \equiv (x-1)(Ax^3 + Bx^2 + Cx + D) \pmod{11},$$

we can solve for A, B, C and D by looking at the various powers of x and equating coefficients modulo 11.

$$\begin{aligned} x^4: &\quad 1 \equiv A \\ x^3: &\quad -1 \equiv -A + B, \text{ giving } B \equiv 0 \\ x^2: &\quad 5 \equiv -B + C, \text{ giving } C \equiv 5 \\ x: &\quad -3 \equiv -C + D, \text{ giving } D \equiv 2 \\ \text{constant}: &\quad -2 \equiv -D, \text{ which provides a check.} \end{aligned}$$

So $x^4 - x^3 + 5x^2 - 3x - 2 \equiv (x-1)(x^3 + 5x + 2) \pmod{11}$.

Now $x \equiv 3 \pmod{11}$ is a solution of $x^3 + 5x + 2 \equiv 0 \pmod{11}$ and so

$$x^3 + 5x + 2 \equiv (x-3)(Ex^2 + Fx + G) \pmod{11}.$$

We can solve this in the same way as above to get $E \equiv 1$, $F \equiv 3$, $G \equiv 3 \pmod{11}$.

Hence

$$x^4 - x^3 + 5x^2 - 3x - 2 \equiv (x-1)(x-3)(x^2 + 3x + 3) \pmod{11}.$$

Aside: In fact, the quadratic $x^2 + 3x + 3$ cannot be factorised modulo 11. In just the same way as a quadratic equation might have 0, 1 or 2 real solutions, it turns out that a quadratic congruence with prime modulus can have 0, 1 or 2 solutions. The quadratic congruence

$$x^2 + 3x + 3 \equiv 0 \pmod{11}$$

has no solutions.

Exercise 4.3

For each of the following polynomials $P(x)$, find all the solutions of $P(x) \equiv 0 \pmod{7}$ by exhaustion, and hence factorise the polynomial modulo 7.

(a) $x^2 + 4x - 4$

(b) $x^3 + 2x^2 + x + 3$

If we apply Proposition 4.5 to one particular polynomial, we get a result that is well worth recording.

Corollary 4.7 *to Proposition 4.5*

For any prime p,

$$x^{p-1} - 1 \equiv (x-1)(x-2)(x-3)\cdots(x-(p-1)) \,(\mathrm{mod}\ p).$$

Proof Consider the congruence

$$x^{p-1} - 1 \equiv 0 \,(\mathrm{mod}\ p).$$

FLT uncovers $p-1$ incongruent solutions since this congruence is satisfied by $x = 1, 2, 3, \ldots, p-1$. Hence, applying Proposition 4.5,

$$x^{p-1} - 1 \equiv (x-1)(x-2)(x-3)\cdots(x-(p-1))Q(x) \,(\mathrm{mod}\ p),$$

where $Q(x)$ has degree $p-1-(p-1) = 0$ modulo p. So $Q(x)$ is a constant polynomial. Inspection of the coefficient of x^{p-1} confirms that $Q(x) \equiv 1 \,(\mathrm{mod}\ p)$, as required. ■

It would be remiss of us not to record what happens when x is taken to be 0 in the corollary.

For any prime p,

$$\begin{aligned} 0 - 1 &\equiv (-1)(-2)\cdots(-(p-1)) \,(\mathrm{mod}\ p) \\ &\equiv (-1)^{p-1}(p-1)! \,(\mathrm{mod}\ p) \\ &\equiv (p-1)! \,(\mathrm{mod}\ p), \quad \text{since } p-1 \text{ is even or} \\ &\qquad\qquad\qquad\qquad p = 2 \text{ in which case } -1 \equiv 1 \,(\mathrm{mod}\ p). \end{aligned}$$

That is,

$$(p-1)! \equiv -1 \,(\mathrm{mod}\ p),$$

which is Wilson's Theorem.

The result of Proposition 4.5 is the backbone of our proof of Lagrange's Theorem. We simply have to draw the right conclusions from our ability to pull out a linear factor corresponding to each solution of the congruence.

Proof of Lagrange's Theorem Suppose to the contrary that a polynomial congruence $P(x) \equiv 0 \,(\mathrm{mod}\ p)$ has degree k but has incongruent solutions $b_1, b_2, \ldots, b_k$ and b_{k+1}. Applying Proposition 4.5 for the first k solutions, we have that

$$P(x) \equiv (x-b_1)(x-b_2)\cdots(x-b_k)P_k(x) \,(\mathrm{mod}\ p),$$

where $P_k(x)$ has degree $k-k$ modulo p. Having degree 0, $P_k(x)$ is a constant polynomial, say, $P_k(x) = a$. Notice that the leading term in $P(x)$ modulo p is ax^k and so, as $P(x)$ has degree k modulo p, $a \not\equiv 0 \,(\mathrm{mod}\ p)$.

The constant a is just the coefficient c_k of Definition 4.2.

Now consider the further solution b_{k+1}:

$$P(b_{k+1}) \equiv (b_{k+1}-b_1)(b_{k+1}-b_2)\cdots(b_{k+1}-b_k)a \equiv 0 \,(\mathrm{mod}\ p).$$

This means that p divides $(b_{k+1}-b_1)(b_{k+1}-b_2)\cdots(b_{k+1}-b_k)a$ and so, p being prime, Euclid's Lemma implies that p divides one of the terms $b_{k+1}-b_i$ or p divides a. But neither of these is possible: p cannot divide

$b_{k+1} - b_i$ as the solutions are incongruent modulo p, and we have seen that $a \not\equiv 0 \pmod{p}$.

What has been contradicted? The only assumption made is that the polynomial congruence of degree k has $k+1$ solutions. This therefore cannot happen and our proof is complete. ■

It is illuminating to see what happens when the modulus is not prime. Consider again the polynomial congruence

$$x^3 - x \equiv 0 \pmod{6},$$

which we found, in Exercise 4.1, to have six solutions. One factorisation of the polynomial $x^3 - x$ is readily spotted:

$$x^3 - x = x(x^2 - 1) = x(x-1)(x+1) \equiv 0 \pmod{6}.$$

Each of the three discovered factors suggests a solution of the congruence: $x \equiv 0$, 1 and $-1 \pmod{6}$ respectively. But $x \equiv 2$, 3 and $4 \pmod{6}$ are also solutions, and the factorisation has not contradicted this. For example, putting $x = 4$ into the factorised polynomial gives

$$4(4-1)(4+1) = 60 \equiv 0 \pmod{6},$$

and so the solution $x \equiv 4 \pmod{6}$ has not gone away!

The problem is that, since 6 is composite, we cannot deduce from $x(x-1)(x+1) \equiv 0 \pmod{6}$ that $x \equiv 0 \pmod{6}$, or $(x-1) \equiv 0 \pmod{6}$ or $(x+1) \equiv 0 \pmod{6}$. The corresponding deduction in Lagrange's Theorem could be made because the modulus was prime, and this was the key to relating factors of the polynomial to solutions of the congruence.

Notice that since $x \equiv 4 \pmod{6}$ is a solution of this congruence, Lemma 4.4 claims that $x - 4$ can be pulled out as a factor on the left-hand side of $x^3 - x \equiv 0 \pmod{6}$. Indeed it can, showing that the one factorisation that we wrote down above is not the only one!

$$\begin{aligned} x^3 - x &\equiv (x-4)(x^2 + 4x + 3) \pmod{6} \\ &\equiv (x-4)(x+1)(x+3) \pmod{6}. \end{aligned}$$

There are other ways of writing $x^3 - x$ as a product of three linear factors to which it is congruent modulo 6.

To complete the theoretical results of the section, we give an interesting consequence of Lagrange's Theorem.

Proposition 4.8

If p is prime and d is a factor of $p-1$ then the congruence

$$x^d - 1 \equiv 0 \pmod{p}$$

has exactly d solutions.

Proof Let $p-1=dr$ for some integer r. If $r=1$ then $d=p-1$ and the result follows from Corollary 4.7. We may assume therefore that $r \geq 2$. Then

$$\begin{aligned} x^{p-1}-1 &= (x^d-1)(x^{d(r-1)}+x^{d(r-2)}+\cdots+x^d+1) \\ &= (x^d-1)Q(x), \end{aligned}$$

where $Q(x)$ has degree $d(r-1)$ modulo p.

Now we know from Corollary 4.7 that $x^{p-1}-1 \equiv 0 \pmod{p}$ has $p-1$ solutions. If a is such a solution then

$$a^{p-1}-1=(a^d-1)Q(a) \equiv 0 \pmod{p},$$

from which Euclid's Lemma informs us that either $(a^d-1) \equiv 0 \pmod{p}$ or $Q(a) \equiv 0 \pmod{p}$. In other words, any solution of $x^{p-1}-1 \equiv 0 \pmod{p}$ is necessarily a solution of either $x^d-1 \equiv 0 \pmod{p}$ or $Q(x) \equiv 0 \pmod{p}$. Hence these latter two congruences have, between them, at least $p-1$ solutions.

Now apply Lagrange's Theorem. The congruence $x^d-1 \equiv 0 \pmod{p}$ has at most d solutions and $Q(x) \equiv 0 \pmod{p}$ has at most $d(r-1)$ solutions. So between them they have at most

$$d+d(r-1)=dr=p-1$$

solutions. It follows that they have, between them, exactly $p-1$ solutions and, in particular, $x^d-1 \equiv 0 \pmod{p}$ has exactly d solutions. ■

4.3 Congruences with composite moduli

This section has been mainly concerned with theoretical results concerning polynomial congruences rather than with developing techniques for actually solving the congruences. For our purposes in this module, we will not have a great need to solve anything other than relatively simple congruences. We have seen that for not too large a prime modulus we can, failing anything better, attack congruences by an exhaustive method involving evaluation of the polynomial over a complete set of residues. To round off the section, we look at a couple of more complicated examples with composite moduli that illustrate general lines of attack.

Consider the polynomial congruence $P(x) \equiv 0 \pmod{m}$, where $m=p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}$. If a is a solution of this congruence then $P(a) \equiv 0 \pmod{m}$ and so $P(a) \equiv 0 \pmod{p_i^{k_i}}$ for each of the prime factors. It follows that any solution of $P(x) \equiv 0 \pmod{m}$ must be a simultaneous solution of the r polynomial congruences $P(x) \equiv 0 \pmod{p_i^{k_i}}$. Conversely, if a is a simultaneous solution of the r polynomial congruences then by repeated application of the corollary to Theorem 1.6 of Chapter 3, a is a solution of $P(x) \equiv 0 \pmod{m}$. So if we solve, individually, each of the congruences involving a single prime, we can then use the Chinese Remainder Theorem to build a solution of the original congruence. Here is a simple example.

Worked Exercise 4.9

Solve $2x^2 + 5x - 3 \equiv 0 \pmod{72}$.

Solution

As $72 = 2^3 \times 3^2$, we first solve $2x^2 + 5x - 3 \equiv 0 \pmod{8}$ and $2x^2 + 5x - 3 \equiv 0 \pmod{9}$. Trying all eight possibilities for the first congruence and all nine for the second, we discover that

$$2x^2 + 5x - 3 \equiv 0 \pmod{8} \text{ has unique solution } x \equiv 5 \pmod{8}$$

and

$$2x^2 + 5x - 3 \equiv 0 \pmod{9} \text{ has solutions } x \equiv 5, 6 \pmod{9}.$$

The congruence $2x^2 + 5x - 3 \equiv 0 \pmod{72}$ therefore has two solutions:

(a) the simultaneous solution of $x \equiv 5 \pmod{8}$ and $x \equiv 5 \pmod{9}$, namely

$$x \equiv 5 \pmod{72}, \text{ and}$$

(b) the simultaneous solution of $x \equiv 5 \pmod{8}$ and $x \equiv 6 \pmod{9}$, namely

$$x \equiv 69 \pmod{72}.$$

Exercise 4.4

Solve $x^2 - x + 4 \equiv 0 \pmod{80}$.

We have now reduced the problem of solving polynomial congruences to the task of solving $P(x) \equiv 0 \pmod{p^k}$ for prime p. In fact, all solutions of this latter congruence can be built up from solutions of $P(x) \equiv 0 \pmod{p}$ in a way illustrated by the following worked exercise.

Worked Exercise 4.10

Solve $x^3 + x + 5 \equiv 0 \pmod{27}$.

Solution

As $27 = 3^3$, we first solve the given congruence modulo 3. Trying $x = 0$, 1 and 2, we find that

$$x^3 + x + 5 \equiv 0 \pmod{3}$$

has the unique solution $x \equiv 2 \pmod{3}$.

Moving up a power of 3, we next investigate

$$x^3 + x + 5 \equiv 0 \pmod{9}.$$

If a is a solution then $a^3 + a + 5$ is divisible by 9 (and therefore by 3), and so a is also a solution of $x^3 + x + 5 \equiv 0 \pmod{3}$. That is, $a \equiv 2 \pmod{3}$, and so the only possibilities are $a \equiv 2$, 5 or $8 \pmod{9}$. Putting in these values:

$$2^3 + 2 + 5 = 15 \equiv 6 \pmod{9},$$
$$5^3 + 5 + 5 = 135 \equiv 0 \pmod{9},$$
$$(-1)^3 + (-1) + 5 = 3 \equiv 3 \pmod{9},$$

and so $x \equiv 5 \pmod{9}$ is the unique solution.

Moving up one more power of 3 and repeating the argument, any solution of

$$x^3 + x + 5 \equiv 0 \pmod{27}$$

must be a solution of $x^3 + x + 5 \equiv 0 \pmod{9}$ and hence must be congruent to 5 modulo 9. So we need to check only the three candidates $x \equiv 5$, 14 and $23 \pmod{27}$:

$$5^3 + 5 + 5 = 135 \equiv 0 \pmod{27},$$
$$14^3 + 14 + 5 = 2763 \equiv 9 \pmod{27},$$
$$(-4)^3 + (-4) + 5 = -63 \equiv 18 \pmod{27}.$$

We conclude that $x \equiv 5 \pmod{27}$ is the unique solution of the given congruence.

The strategy adopted in Worked Exercise 4.10 will work for a general congruence of this form. Each solution of congruence $P(x) \equiv 0 \pmod{p^k}$ is of the form $b + rp^{k-1}$, where b is a solution of the congruence $P(x) \equiv 0 \pmod{p^{k-1}}$ and $0 \leq r < p$.

Exercise 4.5

Solve the following congruences:

(a) $x^2 + x - 7 \equiv 0 \pmod{25}$

(b) $x^2 + 4x - 1 \equiv 0 \pmod{125}$.

Exercise 4.6

(a) Show that $x = 1$ and $x = -2$ both satisfy

$$x^3 + 2x^2 + 6x + 5 \equiv 0 \pmod{7}.$$

Hence factorise $x^3 + 2x^2 + 6x + 5$ modulo 7 and find any further solutions of the congruence.

(b) Solve the following polynomial congruences.

(i) $x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \equiv 0 \pmod{5}$

(ii) $x^{12} - 1 \equiv 0 \pmod{7}$

(iii) $x^5 - x^3 \equiv 0 \pmod{11}$

(iv) $x^5 + x^3 \equiv 0 \pmod{17}$

(c) Solve the following polynomial congruences.

(i) $x^3 + 2x^2 + 3x + 9 \equiv 0 \pmod{35}$

(ii) $x^3 + x^2 + 3x + 1 \equiv 0 \pmod{105}$

(iii) $x^2 - 7x + 31 \equiv 0 \pmod{75}$

5 Fermat's and Wilson's Theorems revisited

This section is not assessed, and is for interest only.

The results of Fermat's Little Theorem and Wilson's Theorem are illustrated in the solutions of two well-known combinatorial problems.

Question 1 *Having enough beads to permit unlimited use of each of n colours, how many different necklaces consisting of p beads can be made, where p is prime?*

Do we understand what is being asked? We will assume that the necklaces are produced by first forming a string of p beads and then fusing the ends of the string. Our first step is therefore to count how many strings of p beads there are. Well, that's fairly straightforward. There are n choices for the first bead, n choices for the second bead, and so on. In all there are n^p strings.

Notice that, of the n^p strings, n strings consist of beads of one colour alone (one such for each colour). Hence there are $n^p - n$ strings involving at least two colours. As the single-colour necklaces are easily counted, we will

put these to one side and concentrate on the ones involving two or more colours.

As one must expect, the $n^p - n$ strings counted will not result in $n^p - n$ different necklaces being produced. For example, the five strings of beads involving colours A, B and C illustrated in Figure 5.1 all result in the same necklace.

The strings in this example are obtained from one another by removing the bead on the left and replacing it on the right of the string. We will call the operation of moving a bead from the left end to the right end a 'recycle'. Two strings that can be obtained from each other by a sequence of recycles clearly correspond to the same necklace.

The five strings in Figure 5.1 are different. But if we consider the string of four beads shown in Figure 5.2, we see that the second recycle produces a string with the same colour scheme, BABA, and that further recycles here produce just two different strings.

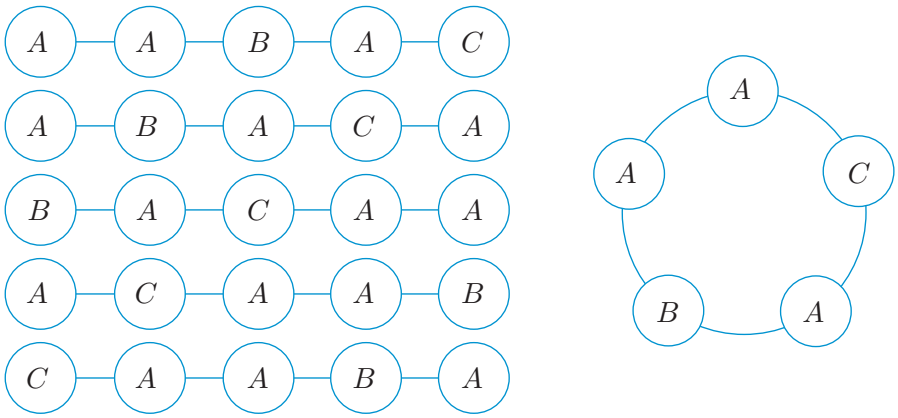


Figure 5.1 Five strings giving the same necklace

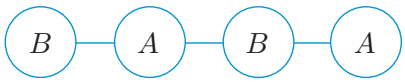


Figure 5.2 String of four beads

So we need to ask how many different strings can be obtained from a string of p beads by recycles. The obvious answer is p, since each of the p beads can occupy the left-end position, unless, as illustrated above for four beads, some fewer number of recycles restores the original colour scheme. We use the Division Algorithm to show that this cannot happen for a prime number of beads.

Let k be the least number of recycles that restores the colour scheme of a string of p beads, where $k > 1$ since we are ignoring strings of a single colour. Dividing p by k, there exist integers q and r such that

$$p = qk + r, \quad \text{where } 0 \leq r < k.$$

Now if k recycles restore the colour scheme then so too do $2k$, $3k$, ..., qk recycles. Furthermore, p recycles certainly restore the colour scheme since each bead returns to its initial position. Therefore after the qkth recycle we have the original colour scheme and only a further r recycles are needed to restore it again. As k is the least number of recycles that can achieve this, we must conclude that $r = 0$. This in turn implies that k divides p and, since p is prime and $k > 1$, we must have $k = p$.

The outcome of this is that the $n^p - n$ strings involving two or more colours are partitioned into disjoint sets of p strings, each set consisting of strings that can be obtained from each other by a sequence of recycles. This means that p divides $n^p - n$, which is Fermat's Little Theorem.

One might be tempted to conclude from this that there are $\dfrac{n^p - n}{p}$ different necklaces involving two or more colours, and hence a grand total of $\dfrac{n^p - n}{p} + n$ different necklaces. Unfortunately there is a further complication to be taken into account. Consider the string shown in Figure 5.3.

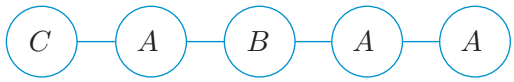


Figure 5.3 String of five beads

It is the first string of Figure 5.1 but with the order reversed (that is, turned over left to right). It gives rise to a further set of five strings that correspond to the same necklace (turned over) as shown in Figure 5.1. So there are really ten different strings that produce this necklace. In general, two different sets of p strings can give rise to the same necklace. We say 'can' because turning a string over does not always produce a new string, as illustrated by the string in Figure 5.4.

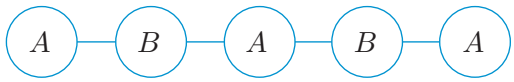


Figure 5.4 A symmetric string

Recycling gives five different strings, but if we then turn it over and recycle we get the same five again. To complete the task we have still to count these 'symmetric' strings. We leave you the task of completing Question 1 as a challenge. For the record, the number of necklaces is

$$\frac{n^p - n}{2p} + \frac{n^{(p+1)/2} - n}{2} + n.$$

Notice that as this is an integer value, the first term still incorporates FLT.

A similar problem leads to a demonstration of Wilson's Theorem.

Question 2 *How many stellated p-gons are there, where p is prime?*

A stellated n-gon is formed by placing n points symmetrically around the circumference of a circle and then joining these points with n straight line segments, crossings being allowed. For $n = 5$ there are twelve stellated pentagons, as shown in Figure 5.5.

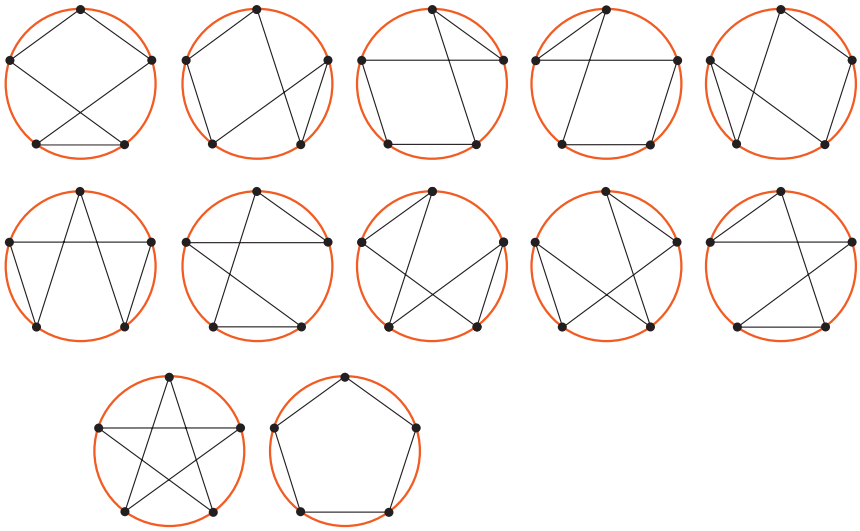


Figure 5.5 The twelve stellated pentagons

Notice that the five pentagons in the first row are congruent, being obtained from each other by rotations through $\frac{2\pi}{5}$. Similarly, the pentagons in the second row are congruent. The remaining two are the *regular* stellated pentagons, having rotation through $\frac{2\pi}{5}$ as a symmetry operation. Hence, if we classify stellated pentagons up to similarity (for we are not concerned about the size) there are two essentially different irregular ones plus the two regular ones.

In the same way we now want to count the number of different stellated p-gons, where $p > 2$ is prime. We will first count how many p-gons can be drawn (corresponding to the twelve pentagons) and then count how many of these are regular ones. The difference will give the irregular ones, which will fall into sets of p congruent p-gons, being obtained from each other by rotation through $\frac{2\pi}{p}$.

To count the total number of stellated p-gons, imagine starting at any point on the circle and drawing the p-gon. There are $p - 1$ choices for the first destination, then $p - 2$ choices for the second destination, and so on. In all, $(p - 1)!$ stellated p-gons can be drawn. But this has counted each p-gon twice, because one ordering of the p vertices visited and the reverse order give the same p-gon. Hence there are $\dfrac{(p-1)!}{2}$ p-gons in all. (This agrees with the discovered 12 for the case $p = 5$.)

When p is not prime this construction will still draw a regular p-gon for each choice of k satisfying $\text{hcf}(k, p) = 1$.

The regular stellated p-gons are formed by choosing any k in the range $1 \leq k \leq p-1$ and joining each point on the circle to the one k places round the circle clockwise from it. But again this counts each regular p-gon twice, because the choices of $k = r$ and $k = p - r$ will draw the same p-gon but visiting the points on the circle in reverse order. Hence there are $\dfrac{(p-1)}{2}$ regular stellated p-gons.

Finally, the $\dfrac{(p-1)!}{2} - \dfrac{(p-1)}{2}$ irregular stellated p-gons are partitioned into congruent sets of p each, as determined by rotation through $\frac{2\pi}{p}$. Hence the total number of different stellated p-gons is

$$\frac{(p-1)! - (p-1)}{2p} + \frac{p-1}{2}.$$

As this is an integer, the first term illustrates that p divides $(p-1)! - (p-1)$, or, in congruence notation, $(p-1)! \equiv -1 \pmod{p}$, which is Wilson's Theorem.

Learning outcomes

We list those topics on which we may set assessment questions to test your understanding of the chapter.

After working through this chapter you should:

(a) Understand the statement, proof and applications of Fermat's Little Theorem and its alternative formulation; recognise which formulation to use in a given scenario.

(b) Know the meaning of the term 'pseudoprime', and its connection with Fermat's Little Theorem.

(c) Be able to apply the Division Algorithm to compute decimal representations of fractions.

(d) Understand the meaning of and be able to compute the 'order' of an integer modulo p; know what it means for the integer 10 to be a 'primitive root' of p; recognise the connection with the decimal representation of rational numbers with prime denominator.

(e) Understand the statement and proof of Wilson's Theorem, and its converse; be able to use Wilson's Theorem to solve more complicated congruences.

(f) Know the definition of the 'degree' of a polynomial congruence.

(g) Understand the statement and proof of Lagrange's Theorem.

(h) Be able to use a variety of techniques to solve polynomial congruences with composite moduli.

Solutions to exercises

Solution to Exercise 1.1

(a) Suppose that $a^p \equiv a \pmod{p}$ for all integers a. Then if $a \not\equiv 0 \pmod{p}$ we can cancel by a to obtain $a^{p-1} \equiv 1 \pmod{p}$, as required.

(b) Suppose that $a^{p-1} \equiv 1 \pmod{p}$ for all a such that $a \not\equiv 0 \pmod{p}$. Then, multiplying by a, $a^p \equiv a \pmod{p}$ for all $a \not\equiv 0 \pmod{p}$. However, if $a \equiv 0 \pmod{p}$ then $a^p \equiv a \equiv 0 \pmod{p}$ and so $a^p \equiv a \pmod{p}$ for all integers a.

Solution to Exercise 1.2

(a) (i) FLT tells us that $5^6 \equiv 1 \pmod{7}$. Therefore

$$5^{20} \equiv (5^6)^3 \times 5^2 \equiv 1^3 \times 25 \equiv 25 \equiv 4 \pmod{7},$$

and the remainder is 4.

(ii) We first replace 37 with its least positive residue before using FLT:

$$37 \equiv 3 \pmod{17}, \quad \text{and so} \quad 37^{37} \equiv 3^{37} \pmod{17}.$$

Now FLT tells us that $3^{16} \equiv 1 \pmod{17}$ and so

$$\begin{aligned} 37^{37} \equiv 3^{37} &\equiv (3^{16})^2 \times 3^5 \equiv 1^2 \times 3^5 \equiv 3^5 \pmod{17} \\ &\equiv 27 \times 9 \equiv 10 \times 9 = 90 \equiv 5 \pmod{17}, \end{aligned}$$

and the remainder is 5.

(b) If $x \equiv a^{p-2}b \pmod{p}$ then $ax \equiv a^{p-1}b \pmod{p}$. Now, since $\text{hcf}(a, p) = 1$, FLT gives $a^{p-1} \equiv 1 \pmod{p}$ and so $ax \equiv b \pmod{p}$, as claimed.

The congruence $5x \equiv 18 \pmod{19}$ has a unique solution modulo 19, which we now know is $x \equiv 5^{17} \times 18 \pmod{19}$. As $5^2 \equiv 6 \pmod{19}$, we have

$$\begin{aligned} x &\equiv 5^{17} \times 18 \pmod{19} \\ &= (5^2)^8 \times 5 \times 18 \pmod{19} \\ &\equiv \left(6^8 \times 5\right) \times (-1) \pmod{19} \\ &\equiv (-2)^4 \times (-5) \pmod{19}, \quad \text{since } 6^2 \equiv -2 \pmod{19}, \\ &\equiv (-3) \times (-5) \pmod{19} \\ &\equiv 15 \pmod{19}. \end{aligned}$$

Solution to Exercise 1.3

(a) As $168 = 3 \times 7 \times 8$, to prove that 168 divides $a^6 - 1$, we will show that $a^6 \equiv 1 \pmod 3$, $a^6 \equiv 1 \pmod 7$ and $a^6 \equiv 1 \pmod 8$. Since 3, 7 and 8 are pairwise coprime, it then follows from Theorem 1.6 of Chapter 3 that $a^6 \equiv 1 \pmod{168}$.

Given that $\text{hcf}(a, 42) = 1$, we know that $\text{hcf}(a, 2) = 1$, $\text{hcf}(a, 3) = 1$ and $\text{hcf}(a, 7) = 1$.

From $\text{hcf}(a, 7) = 1$, FLT gives $a^6 \equiv 1 \pmod 7$.

From $\text{hcf}(a, 3) = 1$, FLT gives $a^2 \equiv 1 \pmod 3$, whereupon

$$a^6 = (a^2)^3 \equiv 1^3 \equiv 1 \pmod 3.$$

Notice that as 8 is not prime we cannot use FLT.

Finally, from $\text{hcf}(a, 2) = 1$ we have that a is odd. But

$$1^2 \equiv 3^2 \equiv 5^2 \equiv 7^2 \equiv 1 \pmod 8,$$

and so $a^2 \equiv 1 \pmod 8$.

Therefore $a^6 = (a^2)^3 \equiv 1^3 \equiv 1 \pmod 8$.

(b) To focus on the units digit of a number amounts to consideration of the number modulo 10. So our task is to show that $a^5 \equiv a \pmod{10}$.

Now FLT gives $a^5 \equiv a \pmod 5$. Moreover, $a^5 \equiv a \pmod 2$, since a^5 is even if, and only if, a is even. As 2 and 5 are coprime it follows that $a^5 \equiv a \pmod{10}$.

What can we now say about $a^{100} \pmod{10}$?

$$a^{100} = (a^5)^{20} \equiv a^{20} = (a^5)^4 \equiv a^4 \pmod{10},$$

and so a^{100} and a^4 have the same units digit.

Solution to Exercise 1.4

(a) $5^{12} \equiv 1 \pmod{13}$ by FLT.

$$5^{14} = 5^{12} \times 5^2 \equiv 1 \times 5^2 = 25 \equiv 12 \pmod{13}.$$

$$\begin{aligned} 1996^{14} &\equiv 7^{14} \pmod{13}, \quad \text{since } 1996 \equiv 7 \pmod{13}, \\ &= 7^{12} \times 7^2 \equiv 1 \times 7^2 \pmod{13}, \quad \text{since } 7^{12} \equiv 1 \pmod{13} \text{ by FLT}, \\ &\equiv 10 \pmod{13}. \end{aligned}$$

(b) (i) FLT gives $3^{16} \equiv 1 \pmod{17}$ and $5^{16} \equiv 1 \pmod{17}$. Therefore

$$3^{50} = (3^{16})^3 \times 3^2 \equiv 1^3 \times 9 \equiv 9 \pmod{17}$$

and

$$5^{50} = (5^{16})^3 \times 5^2 \equiv 1^3 \times 25 \equiv 8 \pmod{17}.$$

Hence $3^{50} + 5^{50} \equiv 9 + 8 \equiv 0 \pmod{17}$, and hence $3^{50} + 5^{50}$ is divisible by 17.

(ii) As 97 is prime, FLT gives $2^{96} \equiv 3^{96} \equiv 1 \pmod{97}$. Therefore

$$\begin{aligned} 2^{100} + 3^{100} &= 2^{96} \times 2^4 + 3^{96} \times 3^4 \equiv 2^4 + 3^4 \pmod{97} \\ &= 16 + 81 \equiv 0 \pmod{97}. \end{aligned}$$

Hence $2^{100} + 3^{100}$ is divisible by 97.

(c) $155 = 5 \times 31$, and FLT gives $7^4 \equiv 1 \pmod 5$ and $7^{30} \equiv 1 \pmod{31}$. Therefore

$$7^{154} = (7^{30})^5 \times 7^4 \equiv 7^4 \equiv 18^2 \equiv 14 \pmod{31}$$

and

$$7^{154} = (7^4)^{38} \times 7^2 \equiv 7^2 \equiv 4 \pmod 5.$$

By the Chinese Remainder Theorem, the simultaneous congruences $x \equiv 14 \pmod{31}$ and $x \equiv 4 \pmod 5$ have a unique solution modulo 155, which can be seen to be $x \equiv 14 \pmod{155}$. So the remainder on dividing 7^{154} by 155 is 14.

(d) As $42 = 2 \times 3 \times 7$, we show that $a^7 \equiv a \pmod 2$, $a^7 \equiv a \pmod 3$ and $a^7 \equiv a \pmod 7$, and then appeal to the corollary to Theorem 1.6 of Chapter 3.

Let a be *any* integer. For each of the primes $p = 2$, 3 and 7 we apply FLT in the form $a^p \equiv a \pmod p$.

$$p = 2: \quad a^7 = (a^2)^3 a \equiv a^3 a = (a^2)^2 \equiv a^2 \equiv a \pmod 2$$
$$p = 3: \quad a^7 = (a^3)^2 a \equiv a^2 a = a^3 \equiv a \pmod 3$$
$$p = 7: \quad a^7 \equiv a \pmod 7$$

Hence, $a^7 \equiv a \pmod{42}$ for all integers a.

(e) By FLT, $p^{q-1} \equiv 1 \pmod q$ and, since q divides q^{p-1},

$$p^{q-1} + q^{p-1} \equiv 1 \pmod q.$$

Similarly

$$p^{q-1} + q^{p-1} \equiv 1 \pmod p.$$

Therefore, by Theorem 1.6 of Chapter 3, since $\text{hcf}(p, q) = 1$,

$$p^{q-1} + q^{p-1} \equiv 1 \pmod{pq}.$$

(f) For odd prime p we have $2^{p-1} \equiv 1 \pmod p$. Therefore

$$(p-1)2^{p-1} + 1 \equiv (p-1) \times 1 + 1 = p \equiv 0 \pmod p$$

and

$$(p-2)2^{p-2} + 1 = p \times 2^{p-2} - 2^{p-1} + 1 \equiv 0 - 1 + 1 \equiv 0 \pmod p.$$

There is no obvious reason why $(p-3)2^{p-3} + 1$ should be divisible by p, and indeed $p = 3$ gives a counter-example:

$$(3-3)2^{3-3} + 1 = 1 \not\equiv 0 \pmod 3.$$

Solution to Exercise 2.1

(a)
$$\begin{aligned} 1 &= 0 \times 15 + 1 \\ 10 &= 0 \times 15 + 10 \\ 100 &= 6 \times 15 + 10 \\ 100 &= 6 \times 15 + 10 \end{aligned}$$

This last equation now repeats indefinitely. So $\frac{1}{15} = 0.0\langle 6 \rangle$.

(b)
$$\begin{aligned} 1 &= 0 \times 41 + 1 \\ 10 &= 0 \times 41 + 10 \\ 100 &= 2 \times 41 + 18 \\ 180 &= 4 \times 41 + 16 \\ 160 &= 3 \times 41 + 37 \\ 370 &= 9 \times 41 + 1 \\ 10 &= 0 \times 41 + 10 \end{aligned}$$

The block of five equations now repeats indefinitely, and so $\frac{1}{41}$ has cycle length 5 and $\frac{1}{41} = 0.\langle 02439 \rangle$.

Solution to Exercise 2.2

FLT tells us that $10^{16} \equiv 1 \pmod{17}$ and so the straightforward approach is to check each smaller positive power of 10 in turn, looking for a least positive residue of 1. Going from one power to the next by multiplying by 10, the successive powers of 10 modulo 17 are

$$10,\ 100 \equiv 15,\ 150 \equiv 14,\ 140 \equiv 4,\ 40 \equiv 6,\ 60 \equiv 9,$$
$$90 \equiv 5,\ 50 \equiv 16,\ 160 \equiv 7,\ 70 \equiv 2,\ 20 \equiv 3,$$
$$30 \equiv 13,\ 130 \equiv 11,\ 110 \equiv 8,\ 80 \equiv 12,\ 120 \equiv 1.$$

So the order of 10 modulo 17 is 16, and the cycle of $\frac{1}{17}$ has length 16.

Solution to Exercise 2.3

We know that $10^{60} \equiv 1 \pmod{61}$ and that c is the smallest positive integer for which $10^c \equiv 1 \pmod{61}$. Suppose that dividing 60 by c gives $60 = cq + r$, where $0 \leq r < c$.

Then $10^{60} = (10^c)^q \times 10^r \equiv 1^q \times 10^r \equiv 10^r \pmod{61}$. But $10^{60} \equiv 1 \pmod{61}$ so $10^r \equiv 1 \pmod{61}$. Therefore $r = 0$, for otherwise the order of 10 modulo 61 would be smaller than c. Hence c divides 60, as claimed.

Solution to Exercise 2.4

Notice that 10 is a primitive root of 17 (from Table 2.2, or from the length of the cycle of $\frac{1}{17}$ being 16 and Proposition 2.3).

$$\begin{aligned} 2 &= 0 \times 17 + 2 \\ 20 &= 1 \times 17 + 3 \\ 30 &= 1 \times 17 + 13 \end{aligned}$$

At this point we know that $\frac{2}{17}$ begins 0.11, which pinpoints the starting point in the cycle of $\frac{1}{17}$. Thus

$$\tfrac{2}{17} = 0.\langle 1176470588235294 \rangle.$$

Similarly

$$\begin{aligned} 9 &= 0 \times 17 + 9 \\ 90 &= 5 \times 17 + 5 \\ 50 &= 2 \times 17 + 16 \end{aligned}$$

shows that $\frac{9}{17}$ begins 0.52, and so we can deduce that

$$\tfrac{9}{17} = 0.\langle 5294117647058823 \rangle.$$

Solution to Exercise 2.5

(a) The system of equations for the decimal fraction of $\frac{1}{21}$ is as follows.

$$\begin{aligned} 1 &= 0 \times 21 + 1 \\ 10 &= 0 \times 21 + 10 \\ 100 &= 4 \times 21 + 16 \\ 160 &= 7 \times 21 + 13 \\ 130 &= 6 \times 21 + 4 \\ 40 &= 1 \times 21 + 19 \\ 190 &= 9 \times 21 + 1 \end{aligned}$$

The equations now cycle. So $\frac{1}{21} = 0.\langle 047619 \rangle$, having a cycle of length 6. As 6 is not a factor of $21 - 1$, Leibniz's claim is disproved.

(b) If $\frac{1}{p}$ has a cycle of length n then $10^n \equiv 1 \pmod{p}$. That is, p divides $10^n - 1$. Hence, $p < 10^n - 1$, confirming that the number of such p is finite.

(c) The prime decomposition of $10^4 - 1$ is

$$10^4 - 1 = 3^2 \times 11 \times 101.$$

If $\frac{1}{p}$ has a cycle of length 4 then $10^4 \equiv 1 \pmod{p}$ and so p must be a factor of $10^4 - 1$. These values are seen to be $p = 3$, 11 and 101. For each of these p, the order of 10 modulo p is a factor of 4, namely 1, 2 or 4. Those p for which the order is less than 4 must satisfy $10^2 \equiv 1 \pmod{p}$, which happens only for p a factor of $10^2 - 1 = 99$; that is, $p = 3$ or 11. We conclude that the only prime p for which $\frac{1}{p}$ has a cycle of length 4 is 101.

In the same way, those p for which $\frac{1}{p}$ has a cycle of length 6 are those prime factors of $10^6 - 1$ that are not factors of either $10^2 - 1$ or $10^3 - 1$, that is, any factor of $3^3 \times 7 \times 11 \times 13 \times 37$ that is not a factor of either $3^2 \times 11$ or $3^3 \times 37$. The only primes satisfying this condition are 7 and 13.

(d) (i) If all cards return to their original position after n shuffles then

$$2^n x \equiv x \,(\text{mod } 53), \text{ for all } 1 \leq x \leq 52.$$

As $\text{hcf}(x, 53) = 1$ we may cancel x to obtain

$$2^n \equiv 1 \,(\text{mod } 53).$$

So n has to be a multiple of the order of 2 modulo 53, and indeed if n shuffles is the smallest number for which this occurs, n is the order of 2 modulo 53.

FLT tells us that $2^{52} \equiv 1 \,(\text{mod } 53)$ so, by Proposition 2.4, the n we seek is a factor of 52. Checking $n = 2$, 4, 13 and 26 we find

$$2^2 = 4,$$
$$2^4 = 16,$$
$$2^{13} = 2^6 \times 2^6 \times 2 \equiv 11 \times 11 \times 2 \equiv 30 \,(\text{mod } 53),$$
$$2^{26} = (2^{13})^2 \equiv 30^2 \equiv 52 \,(\text{mod } 53).$$

Therefore, as no smaller power of 2 is congruent to 1 modulo 53, the order of 2 modulo 53 is 52, and 52 shuffles are needed to restore the original order.

(ii) When the number of cards in the pack goes up to 54, we have to find the smallest natural number n for which $2^n \equiv 1 \,(\text{mod } 55)$. We note that 55 is not prime and so we are not able to start from $2^{54} \equiv 1 \,(\text{mod } 55)$. However, noting that $55 = 5 \times 11$, FLT gives

$$2^4 \equiv 1 \,(\text{mod } 5) \text{ and } 2^{10} \equiv 1 \,(\text{mod } 11).$$

As $\text{lcm}(4, 10) = 20$, we have

$$2^{20} = (2^4)^5 \equiv 1 \,(\text{mod } 5) \text{ and } 2^{20} = (2^{10})^2 \equiv 1 \,(\text{mod } 11).$$

Since 5 and 11 are coprime, the corollary to Theorem 1.6 of Chapter 3 gives $2^{20} \equiv 1 \,(\text{mod } 55)$, and we have one solution to our problem: 20 shuffles will restore the original order of the pack of 54 cards.

The fact that 55 is not a prime does not affect the necessity for n to satisfy $2^n \equiv 1 \,(\text{mod } 55)$, as $2^n x \equiv x \,(\text{mod } 55)$ has to hold for *all* x, including $x = 1$.

If c is the smallest positive integer such that $2^c \equiv 1 \,(\text{mod } 55)$ then c must divide 20. Why? Suppose $20 = qc + r$, where $0 \leq r < c$. Then, as in the proof of Proposition 2.4,

$$a^{20} \equiv a^{qc+r} \equiv (a^c)^q \times a^r \equiv 1^q a^r \,(\text{mod } 55)$$

showing that $a^r \equiv 1 \,(\text{mod } 55)$ with $r < c$. Hence $r = 0$ and so c must divide 20.

However,

$$2^2 = 4,$$
$$2^4 = 16,$$
$$2^5 = 32,$$
$$2^{10} = 2^6 \times 2^4 \equiv 9 \times 16 = 144 \equiv 34 \pmod{55},$$

which shows that no smaller power of 2 is congruent to 1 modulo 55. So 20 is the least number of shuffles that restores the original order.

Solution to Exercise 3.1

(a) Wilson's Theorem assures us that 19 divides $18! + 1$. For any prime $p < 19$, p divides $18!$ and so p cannot divide $18! + 1$. Hence 19 is the smallest prime dividing $18! + 1$.

(b) Wilson's Theorem tells us that 31 divides $30! + 1$. Now

$$30! + 1 = 30 \times 29! + 1 \equiv (-1)29! + 1 = 1 - 29! \pmod{31},$$

and so 31 also divides $29! - 1$. Furthermore, 31 must be the smallest prime with this property because any prime $p < 31$ divides $29!$ and so cannot divide $29! - 1$.

Solution to Exercise 3.2

(a) Worked Exercise 3.3 showed that for any prime $p = 4k + 1$, the congruence $x^2 + 1 \equiv 0 \pmod{4k+1}$ has $x \equiv (2k)! \pmod{4k+1}$ as a solution. For $k = 7$, the congruence $x^2 + 1 \equiv 0 \pmod{29}$ has solution $x \equiv 14! \pmod{29}$. Working modulo 29, we can simplify the calculation of $14!$ by pairing the terms appropriately:

This, of course, is not a very efficient method of solving the congruence. Its inclusion is to highlight the construction of Worked Exercise 3.3.

$$\begin{aligned} 14! &= 1 \times (2 \times 14) \times (3 \times 10) \times (4 \times 7) \times (5 \times 6) \times (8 \times 11) \times (9 \times 13) \times 12 \\ &\equiv 1 \times (-1) \times 1 \times (-1) \times 1 \times 1 \times 1 \times 12 \pmod{29} \\ &\equiv 12 \pmod{29}. \end{aligned}$$

Since $(-x)^2 = x^2$, $x \equiv -12 \equiv 17 \pmod{29}$ is a second solution.

In fact $x \equiv 12, 17 \pmod{29}$ are the only solutions of this congruence, as you will see from the work of the next section.

(b) Let us first illustrate the proof for the particular case $p = 11$. Working modulo 11, we have

$$2 \equiv -9,\ 4 \equiv -7,\ 6 \equiv -5,\ 8 \equiv -3 \text{ and } 10 \equiv -1.$$

Therefore,

$$\begin{aligned} -1 &\equiv 10! \pmod{11}, \quad \text{by Wilson's Theorem,} \\ &\equiv 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \pmod{11} \\ &\equiv 1 \times 3 \times 5 \times 7 \times 9 \times (-9) \times (-7) \times (-5) \times (-3) \times (-1) \pmod{11} \\ &\equiv (-1)^5 \times 1^2 \times 3^2 \times 5^2 \times 7^2 \times 9^2 \pmod{11}. \end{aligned}$$

Multiplying through the final congruence by $(-1)^5$ gives

$$1^2 \times 3^2 \times 5^2 \times 7^2 \times 9^2 \equiv (-1)^6 \pmod{11},$$

which is one of the required formulas. The second comes quickly from the observation that from $2 \equiv -9 \pmod{11}$ we have $2^2 \equiv 9^2 \pmod{11}$ and similarly

$$4^2 \equiv 7^2,\ 6^2 \equiv 5^2,\ 8^2 \equiv 3^2 \text{ and } 10^2 \equiv 1^2 \pmod{11}.$$

Substitution gives

$$2^2 \times 4^2 \times 6^2 \times 8^2 \times 10^2 \equiv (-1)^6 \pmod{11}.$$

The same argument works for any odd prime p:

$$2 \equiv -(p-2),\ 4 \equiv -(p-4),\ \ldots,\ (p-1) \equiv -1 \pmod{p},$$

and so

$$\begin{aligned} -1 &\equiv (p-1)! \pmod{p} \\ &= (1 \times 3 \times 5 \times \cdots \times (p-2)) \times (2 \times 4 \times 6 \times \cdots \times (p-1)) \pmod{p} \\ &\equiv (1 \times 3 \times \cdots \times (p-2)) \times (-(p-2) \times (-(p-4)) \times \cdots \times (-1)) \pmod{p} \\ &\equiv (-1)^{(p-1)/2} \times 1^2 \times 3^2 \times \cdots \times (p-2)^2 \pmod{p}. \end{aligned}$$

Multiplying by $(-1)^{(p-1)/2}$ gives

$$1^2 \times 3^2 \times \cdots \times (p-2)^2 \equiv (-1)^{(p+1)/2} \pmod{p}.$$

For each k on the left of this congruence, we can replace k^2 with $(p-k)^2$ to get the alternative formula

$$2^2 \times 4^2 \times \cdots \times (p-1)^2 \equiv (-1)^{(p+1)/2} \pmod{p}.$$

Solution to Exercise 3.3

(a) The successive powers of 2 modulo 29 are given in the table below.

To get from one power to the next we just 'double modulo 29'.

n	2^n	n	2^n	n	2^n	n	2^n
1	2	8	24	15	27	22	5
2	4	9	19	16	25	23	10
3	8	10	9	17	21	24	20
4	16	11	18	18	13	25	11
5	3	12	7	19	26	26	22
6	6	13	14	20	23	27	15
7	12	14	28	21	17	28	1

Note that 2^n takes all 28 non-zero values modulo 29, confirming that the order of 2 modulo 29 is 28. Now, if $m + n = 28$ then $2^m 2^n = 2^{m+n} = 2^{28} \equiv 1 \pmod{29}$. For example, taking $m = 8$ and $n = 20$,

$$2^8 \times 2^{20} \equiv 24 \times 23 \equiv (-5) \times (-6) = 30 \equiv 1 \pmod{29}.$$

In all there are 13 such pairs, 1 with 27, 2 with 26, ... and 13 with 15.

The value $n = 14$ (for which $2^{14} \equiv 28 \equiv -1 \pmod{29}$) is omitted because it pairs with itself. The pairs are

$$\begin{aligned} 2 \times 15 &\equiv 4 \times 22 \equiv 8 \times 11 \equiv 16 \times 20 \equiv 3 \times 10 \equiv 6 \times 5 \equiv 12 \times 17 \\ &\equiv 24 \times 23 \equiv 19 \times 26 \equiv 9 \times 13 \equiv 18 \times 21 \equiv 7 \times 25 \equiv 14 \times 27 \\ &\equiv 1 \pmod{29}. \end{aligned}$$

Multiplying all these thirteen pairs of numbers together gives

$$27! \equiv 1 \pmod{29},$$

and multiplying this through by 28 we reach

$$28! \equiv 28 \equiv -1 \pmod{29},$$

completing the illustration of the proof.

(b) For any integer a and prime p, FLT gives $a^p \equiv a \pmod{p}$ and Wilson's Theorem gives $(p-1)! \equiv -1 \pmod{p}$. Therefore

$$a^p + a \times (p-1)! \equiv a + a \times (-1) \equiv 0 \pmod{p}$$

and

$$a + a^p \times (p-1)! \equiv a + a \times (-1) \equiv 0 \pmod{p}.$$

(c) The result is certainly true for $p = 2$, and so we assume that p is odd. Since

$$1 + 2 + 3 + \cdots + (p-1) = \frac{p(p-1)}{2},$$

and p and $\dfrac{p-1}{2}$ are coprime integers, it suffices to show that

$$(p-1)! \equiv p - 1 \pmod{p}$$

and

$$(p-1)! \equiv p - 1 \pmod{\frac{p-1}{2}}.$$

The first of these is precisely Wilson's Theorem, and the second simply says that $0 \equiv 0$ since $\dfrac{p-1}{2}$ is a factor of $p-1$, and so

$$(p-1)! \equiv p - 1 \pmod{\frac{p(p-1)}{2}}.$$

Solution to Exercise 4.1

(a) Trying $x = 0$, 1, 2, 3, 4 and 5, we find that $x^3 \equiv x \pmod{6}$ in each case and so the congruence $x^3 - x \equiv 0 \pmod{6}$ has six solutions, namely $x \equiv 0, 1, 2, 3, 4, 5 \pmod{6}$.

(b) The straightforward approach is to test the result as x runs through a complete set of residues. However, recalling that $x^2 - 1 \equiv 0 \pmod{8}$ whenever x is odd, it follows that

See the solution to Exercise 1.3(a) in this chapter.

$$x^3 - x = x(x^2 - 1) \equiv 0 \pmod{8} \text{ for } x \equiv 1, 3, 5, 7 \pmod{8}.$$

For x even, $0^3 - 0 \equiv 0$, $2^3 - 2 \equiv 6$, $4^3 - 4 \equiv 4$ and $6^3 - 6 \equiv 2 \pmod{8}$.

Hence there are five solutions in all, $x \equiv 0, 1, 3, 5, 7 \pmod{8}$.

(c) The values of the polynomial for a complete set of residues are given below.

x	$x^2+x+1 \pmod 7$
0	1
1	3
2	$7 \equiv 0$
3	$13 \equiv 6$
4	$21 \equiv 0$
5	$31 \equiv 3$
6	$43 \equiv 1$

The numbers in the calculations here could have been kept smaller by using the set of least absolute residues modulo 7.

There are two solutions, namely $x \equiv 2, 4 \pmod 7$.

(d) First, we simplify the congruence by reducing the coefficients modulo 7 to get the equivalent congruence $2x^2+2x+1 \equiv 0 \pmod 7$. As x takes values 0, 1, 2, 3, 4, 5 and 6, $2x^2+2x+1$ takes respective values $\pmod 7$ 1, 5, 6, 4, 6, 5 and 1. Therefore this congruence has no solutions.

Solution to Exercise 4.2

(a) This has degree 6. You may have noticed that this congruence can be reduced by FLT to an equivalent congruence of lower degree (0, in fact), but this is not immediately pertinent.

(b) Since $10 \equiv 0 \pmod 5$, the leading term modulo 5 is $3x$, and so the congruence has degree 1.

(c) Since $15 \equiv 0 \pmod 5$ and $-10 \equiv 0 \pmod 5$, the leading term modulo 5 is the constant 2, and so the congruence has degree 0.

Solution to Exercise 4.3

(a) Evaluating x^2+4x-4 for $x = 0, \pm1, \pm2$ and ±3 gives two solutions, $x \equiv -1$ and $x \equiv -3$. Pulling out the factor $(x+1)$ corresponding to the first of these solutions,

$$x^2+4x-4 \equiv (x+1)(Ax+B) \pmod 7$$

produces $A = 1$, $B = 3$ and the factorisation

$$x^2+4x-4 \equiv (x+1)(x+3) \pmod 7.$$

The factors $x+1$ and $x+3$ could equally well be written as $x-6$ and $x-4$ respectively.

The second factor is the one expected, corresponding to the solution $x \equiv -3 \pmod 7$.

(b) By exhaustion, we find that $x^3+2x^2+x+3 \equiv 0 \pmod 7$ has just the two solutions $x \equiv 1, 2 \pmod 7$.

Pulling out the factor $(x-1)$ gives

$$x^3+2x^2+x+3 \equiv (x-1)(x^2+3x+4) \pmod 7,$$

and then pulling out the factor $(x-2)$ gives

$$\begin{aligned} x^3+2x^2+x+3 &\equiv (x-1)(x-2)(x+5) \pmod 7 \\ &\equiv (x-1)(x-2)^2 \pmod 7. \end{aligned}$$

Solution to Exercise 4.4

As $80 = 2^4 \times 5$, we solve the congruences $x^2 - x + 4 \equiv 0 \pmod{16}$ and $x^2 - x + 4 \equiv 0 \pmod{5}$. By exhaustion,

$$x^2 - x + 4 \equiv 0 \pmod{16}$$

has solutions $x \equiv -3, 4 \pmod{16}$, and

$$x^2 - x + 4 \equiv 0 \pmod{5}$$

has the unique solution $x \equiv 3 \pmod{5}$.

Hence $x^2 - x + 4 \equiv 0 \pmod{80}$ has two solutions. By solving simultaneously,

$x \equiv -3 \pmod{16}$ and $x \equiv 3 \pmod{5}$ yield $x \equiv 13 \pmod{80}$

and

$x \equiv 4 \pmod{16}$ and $x \equiv 3 \pmod{5}$ yield $x \equiv 68 \pmod{80}$.

Solution to Exercise 4.5

(a) By trial and error we find that $x^2 + x - 7 \equiv 0 \pmod{5}$ has solutions $x \equiv 1, 3 \pmod{5}$. So the only candidates for solution of $x^2 + x - 7 \equiv 0 \pmod{25}$ are:

from $x \equiv 1 \pmod{5}$, $x \equiv 1, 6, 11, 16$ and $21 \pmod{25}$

and

from $x \equiv 3 \pmod{5}$, $x \equiv 3, 8, 13, 18$ and $23 \pmod{25}$.

Of these,

$$11^2 + 11 - 7 = 125 \equiv 0 \pmod{25}$$

and

$$13^2 + 13 - 7 = 175 \equiv 0 \pmod{25}$$

are the only solutions. Hence $x^2 + x - 7 \equiv 0 \pmod{25}$ has solutions $x \equiv 11, 13 \pmod{25}$.

As 5 is prime, we know by Lagrange's Theorem there are at most two solutions; once we have found them we can stop.

This method still involves much computation, but whereas exhaustion would involve testing 25 values modulo 25, we have reduced this number to 10.

(b) The congruence $x^2 + 4x - 1 \equiv 0 \pmod{5}$ has unique solution $x \equiv 3 \pmod{5}$. The candidates for solution of $x^2 + 4x - 1 \equiv 0 \pmod{25}$ are therefore $x \equiv 3, 8, 13, 18$ and $23 \pmod{25}$.

But

$$3^2 + 12 - 1 = 20 \equiv 20 \pmod{25},$$
$$8^2 + 32 - 1 = 95 \equiv 20 \pmod{25},$$
$$13^2 + 52 - 1 = 220 \equiv 20 \pmod{25},$$
$$(-7)^2 - 28 - 1 = 20 \equiv 20 \pmod{25},$$
$$(-2)^2 - 8 - 1 = -5 \equiv 20 \pmod{25}.$$

Hence $x^2 + 4x - 1 \equiv 0 \pmod{25}$ has no solutions, and from this $x^2 + 4x - 1 \equiv 0 \pmod{125}$ cannot have any solutions.

This time we have saved a lot of computation!

Solution to Exercise 4.6

(a) When $x = 1$,

$$x^3 + 2x^2 + 6x + 5 = 1 + 2 + 6 + 5 = 14 \equiv 0 \pmod 7,$$

and when $x = -2$,

$$x^3 + 2x^2 + 6x + 5 = -8 + 8 - 12 + 5 = -7 \equiv 0 \pmod 7.$$

So both are solutions and it follows that

$$x^3 + 2x^2 + 6x + 5 \equiv (x - 1)(x + 2)(Ax + B) \pmod 7.$$

A little algebra reveals that $A \equiv B \equiv 1 \pmod 7$ and consequently $x + 1$ is also a factor. That gives a third solution, $x \equiv -1 \pmod 7$. As Lagrange's Theorem guarantees that this congruence of degree 3 cannot have more than three solutions, we conclude that $x \equiv 1, -2, -1 \pmod 7$ are all the solutions.

The least positive residue solutions are $x \equiv 1, 5, 6 \pmod 7$.

(b) (i) Since $x \equiv 0 \pmod 5$ is not a solution, we may assume that $\text{hcf}(5, x) = 1$ so that $x^4 \equiv 1 \pmod 5$. Substituting this into the polynomial congruence produces

$$2(x^3 + x^2 + x + 1) \equiv 0 \pmod 5.$$

Trial and error reveals that $x \equiv 2$, 3 and $4 \pmod 5$ are the only three solutions.

(ii) $x^{12} - 1 = (x^6 - 1)(x^6 + 1) \equiv 0 \pmod 7$

$x \equiv 0 \pmod 7$ is not a solution, but for all other x we know from FLT that $x^6 - 1 \equiv 0 \pmod 7$. So the congruence has six solutions, namely $x \equiv 1$, 2, 3, 4, 5 and $6 \pmod 7$.

(iii) $x^5 - x^3 = x^3(x^2 - 1) \equiv 0 \pmod{11}$

If $x^3 \equiv 0 \pmod{11}$ then 11 divides x^3. But that can happen only when 11 divides x.

$x^3 \equiv 0 \pmod{11}$ has the unique solution $x \equiv 0 \pmod{11}$, whilst $x^2 - 1 \equiv 0 \pmod{11}$ has at most two solutions (by Lagrange's Theorem), which can be seen to be $x \equiv \pm 1 \pmod{11}$. So there are three solutions, $x \equiv 0$, 1 and $10 \pmod{11}$.

(iv) $x^5 + x^3 = x^3(x^2 + 1) \equiv 0 \pmod{17}$

$x^3 \equiv 0 \pmod{17}$ has the unique solution $x \equiv 0 \pmod{17}$, whilst $x^2 + 1 \equiv 0 \pmod{17}$ has at most two solutions (by Lagrange's Theorem). Writing the latter in the equivalent form $x^2 \equiv 16 \pmod{17}$, the solutions are seen to be $x \equiv \pm 4 \pmod{17}$. So there are three solutions, $x \equiv 0$, 4 and $13 \pmod{17}$.

(c) (i) $x^3 + 2x^2 + 3x + 9 \equiv 0 \pmod{5 \times 7}$

We solve the congruence modulo 5 and modulo 7 and then look for simultaneous solutions.

$$x^3 + 2x^2 + 3x + 4 \equiv 0 \pmod 5$$

has a unique solution (found by exhaustion) $x \equiv 1 \pmod 5$.

$$x^3 + 2x^2 + 3x + 2 \equiv 0 \pmod 7$$

has solutions $x \equiv 3, 6 \pmod 7$.

$x \equiv 1 \pmod 5$ and $x \equiv 3 \pmod 7$ if, and only if, $x \equiv 31 \pmod{35}$.

$x \equiv 1 \pmod 5$ and $x \equiv 6 \pmod 7$ if, and only if, $x \equiv 6 \pmod{35}$.

So there are two solutions, $x \equiv 6, 31 \pmod{35}$.

(ii) $x^3 + x^2 + 3x + 1 \equiv 0 \pmod{3 \times 5 \times 7}$

Proceeding as in part (i), we first solve the congruence for each of the moduli 3, 5 and 7. However, this congruence has no solutions modulo 5 (as discovered by trying each of the values 0, ± 1, ± 2 for x). Hence the congruence has no solutions modulo 105.

(iii) As $75 = 3 \times 25$, we first solve $x^2 - 7x + 31 \equiv 0 \pmod 3$, then solve $x^2 - 7x + 31 \equiv 0 \pmod{25}$ and finally find simultaneous solutions.

Working modulo 3:

$$x^2 - 7x + 31 \equiv x^2 - x + 1 \equiv 0 \pmod 3,$$

which is seen to have the unique solution $x \equiv 2 \pmod 3$.

To solve $x^2 - 7x + 31 \equiv 0 \pmod{25}$ we first find the solutions, if any, modulo 5. As

$$x^2 - 7x + 31 \equiv x^2 - 2x + 1 = (x - 1)^2 \equiv 0 \pmod 5,$$

this has the unique solution $x \equiv 1 \pmod 5$. It follows that the only candidates for solutions modulo 25 are

$$x \equiv 1, 6, 11, 16, 21 \pmod{25}.$$

A little arithmetic confirms that each of these five is a solution.

Hence the solutions of the original congruence are found by solving $x \equiv 2 \pmod 3$ simultaneously with each of these five values modulo 25. Five solutions emerge:

$$x \equiv 11, 26, 41, 56, 71 \pmod{75}.$$